CONTINUOUS-IN-TIME BUBBLING AND PROGRESS TOWARDS SOLITON RESOLUTION CONJECTURE FOR THE ENERGY-CRITICAL NONLINEAR HEAT FLOW

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Abstract. We show that any finite energy solution of the energy-critical nonlinear heat flow in dimensions $d \geq 3$ asymptotically resolves into a sum of solitons, possibly time-dependent, a radiation term, and an error term that vanishes in the energy space. As a consequence, when the initial data has finite energy and is non-negative, we settle the Soliton Resolution Conjecture for all dimensions $d \geq 3$.

1. Introduction

1.1. **Problem Setting.** In this work, we study the long-term behavior of solutions to the energy-critical nonlinear heat flow in dimension $d \geq 3$:

$$\begin{aligned} \partial_t u &= \Delta u + |u|^{p-1} u \\ u(0,x) &= u_0(x) \in \dot{H}^1(\mathbb{R}^d), \end{aligned} \tag{1.1}$$

where $p := \frac{d+2}{d-2}$. This model arises as the negative gradient flow of the following nonlinear energy functional:

$$E(u) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \mathrm{d}x - \frac{1}{p+1} \int_{\mathbb{R}^d} |u(x)|^{p+1} \mathrm{d}x, \tag{1.2}$$

which appears naturally in the study of extremizers of the Sobolev inequality and, more generally, is connected to the Yamabe problem on the sphere via stereographic projection. The local well-posedness of (1.1) in $\dot{H}^1$-norm is classical and was initiated by Weissler in [Wei79, Wei80], with further contributions by Giga [Gig86], Ni–Sacks [NS85], and Brezis-Cazenave [BC96]. Observe that the solutions of (1.1) are invariant under translations and parabolic scaling,

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{-\frac{d-2}{2}} u\left(t/\lambda^2, x/\lambda\right), \quad \lambda > 0.$$

Since the nonlinear energy is invariant under these symmetries, i.e., $E(u) = E(u_\lambda)$, the equation (1.1) is energy-critical. Testing (1.1) against $\partial_t u$ and integrating by parts we observe the formal energy identity

$$E(u(T)) + \int_0^T \|\partial_t u\|_{L^2}^2 \, \mathrm{d}t = E(u(0)), \tag{1.3}$$

for each $T > 0$. In particular, this implies that the nonlinear energy is non-increasing along the flow. Any function $W : \mathbb{R}^d \to \mathbb{R}$ solving the elliptic PDE

$$\Delta W + |W|^{p-1} W = 0 \tag{1.4}$$

is a stationary solution and will often be referred to as a bubble or soliton of (1.1).

1.2. **Statement of the Main result.** To state our main Theorem 1.6, we first define a notion of *scale* and *center* of a non-zero stationary solution. Let $S > 0$ in $\mathbb{R}^d$ be the sharp constant for the Sobolev inequality,

$$\|u\|_{L^{p+1}} \leq S\|\nabla u\|_{L^2}$$

for all $u \in \dot{H}^1$. Then observe that for any non-zero stationary solution $W \in \dot{H}^1$ we have

$$\int_{\mathbb{R}^d} |\nabla W|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |W|^{p+1} \mathrm{d}x.$$

By the variational characterization of the Sobolev inequality, the best constant (or equality) is attained by a positive stationary solution W', i.e.,

$$\int_{\mathbb{R}^d} |W'|^{p+1} \mathrm{d}x = S^{p+1} \left(\int_{\mathbb{R}^d} |\nabla W'|^2 \mathrm{d}x \right)^{(p+1)/2} = \int_{\mathbb{R}^d} |\nabla W'|^2 \mathrm{d}x$$

implying that for any positive stationary solution we have

$$\int_{\mathbb{R}^d} |\nabla W'|^2 \mathrm{d}x = S^{-d}.$$

Therefore, for any sign-changing stationary solution, we have

$$\int_{\mathbb{R}^d} |\nabla W|^2 \mathrm{d}x = \int_{\mathbb{R}^d} |\nabla W^+|^2 \mathrm{d}x + \int_{\mathbb{R}^d} |\nabla W^-|^2 \mathrm{d}x > 2S^{-d}.$$

In particular, we deduce that any non-zero stationary solution $W \in \dot{H}^1$ satisfies

$$\|\nabla W\|_{L^2}^2 \geq S^{-d}.$$

Denote $\bar{E}_* := S^{-d}$ as the minimal energy of any non-zero stationary solution of (1.4), and in general, let $\bar{E}(u) := \|\nabla u\|_{L^2}^2$ for any $u \in \dot{H}^1$. Thus, given any non-zero stationary solution $W : \mathbb{R}^d \to \mathbb{R}$, we define its scale and center as follows:

Definition 1.1 (Scale of a stationary solution)**.** Let $\gamma_0 \in (0, \bar{E}_*/2)$. Then the scale associated to a non-trivial stationary solution W, denoted by $\lambda(W;\gamma_0)$, is defined by

$$\lambda(W;\gamma_0) := \inf\{\lambda \in (0,\infty) \mid \exists\, a \in \mathbb{R}^d \text{ such that } \bar{E}(W; B(a,\lambda)) \geq \bar{E}(W) - \gamma_0\}.$$

Definition 1.2 (Center of a stationary solution)**.** Let $\gamma_0 \in (0, \bar{E}_*/2)$ and let $\lambda(W;\gamma_0)$ be the scale of a non-zero stationary solution W. Then the center, denoted by $a(W;\gamma_0) \in \mathbb{R}^d$, is defined as

$$\bar{E}(W; B(a(W;\gamma_0), \lambda(W;\gamma_0))) \geq \bar{E}(W) - \gamma_0.$$

These quantities are well-defined as we will later prove in Lemma 2.1. Since our main result says that finite energy solutions of (1.1) eventually approach a sum of stationary solutions, it will be convenient to define their sum, which we will often refer to as a multi-bubble configuration.

Definition 1.3 (Multi-bubble configuration)**.** Let $K \in \{0, 1, 2, \dots\}$. A K-multi-bubble configuration is the sum

$$\mathbf{W}(x) = \sum_{j=1}^{K} W_j(x),$$

where $W_j : \mathbb{R}^d \to \mathbb{R}$ are smooth non-zero stationary solution. By convention if $K = 0$ then $\mathbf{W} \equiv 0$. To emphasize the dependence of $\mathbf{W}$ on the collection $\{W_j\}_{j=1}^K$, we will occasionally write $\mathbf{W} = \mathbf{W}(\vec{W})$, where $\vec{W} = (W_1, \dots, W_K)$.

Next, we quantify the distance of a function to some multi-bubble configuration.

Definition 1.4 (Localized distance to a multi-bubble configuration)**.** Given,

(1) some scales $\xi, \rho, \nu \in (0, \infty)$, such that $\xi \le \rho \le \nu$;
(2) a map $u : [0, T_+) \times B(y, \nu) \to \mathbb{R}$, where $T_+ > 0$ and $\gamma_0 \in (0, \bar{E}_*/2)$;
(3) a non-negative integer $K \in \mathbb{N}$ and non-zero stationary solutions $W_1, \dots, W_K$ with centers $a(W_j; \gamma_0) \in B(y, \xi)$ and scales $\lambda(W_j; \gamma_0) \in (0, \infty)$ for each $j \in \{1, \dots, K\}$;
(4) collection of radii $\vec{\nu} = (\nu, \nu_1, \dots, \nu_K) \in (0, \infty)^{K+1}$ such that $B(a(W_j), \nu_j) \subset B(y, \xi)$ and smaller scales $\vec{\xi} = (\xi, \xi_1, \dots, \xi_K) \in (0, \infty)^{K+1}$ such that $\xi_j < \lambda(W_j; \gamma_0)$ for each $j \in \{1, \dots, K\}$.

Then the localized distance is defined as

$$\begin{aligned}\mathbf{d}_{\gamma_0}(u(t), \mathbf{W}; B(y, \rho); \vec{\nu}, \vec{\xi}) &:= \bar{E}\big(u - \mathbf{W}(\vec{W})); B(y, \rho)\big) + \bar{E}(u; B(y, \nu) \setminus B(y, \xi)) \\ &\quad + \frac{\xi}{\rho} + \frac{\rho}{\nu} + \sum_{j \neq k} \left(\frac{\lambda(W_j)}{\lambda(W_k)} + \frac{\lambda(W_k)}{\lambda(W_j)} + \frac{|a(W_j) - a(W_k)|}{\lambda(W_j)} \right)^{-(d-2)/2} \\ &\quad + \sum_j \Big(\frac{\lambda(W_j)}{\operatorname{dist}(a(W_j), \partial B(y, \xi))} + \frac{\lambda(W_j)}{\nu_j} + \frac{\xi_j}{\lambda(W_j)} \Big) \\ &\quad + \sum_j \sum_{k \in \mathcal{I}_j} \frac{\xi_j}{\operatorname{dist}(a(W_k), \partial B(a(W_j), \nu_j))}.\end{aligned}$$

Minimizing over all the parameters in the above definition yields,

Definition 1.5 (Localized multi-bubble proximity function)**.** Given, $y \in \mathbb{R}^d$, $\rho \in (0, \infty)$, $u : [0, T_+) \times B(y, \rho) \to \mathbb{R}$, where $T_+ > 0$ and $\gamma_0 \in (0, \bar{E}_*/2)$, define

$$\boldsymbol{\delta}_{\gamma_0}(u(t); B(y, \rho)) := \inf_{\mathbf{W}, \vec{\nu}, \vec{\xi}} \mathbf{d}_{\gamma_0}(u, \mathbf{W}; B(y, \rho); \vec{\nu}, \vec{\xi})$$

where the infimum above is taken over all possible K-multi bubble configurations for any non-negative integer K, over all parameters $\vec{\nu} \in (0, \infty)^{K+1}$ and $\vec{\xi} \in (0, \infty)^{K+1}$ as in Definition 1.4. Since we will fix γ_0 later, we drop the subscript involving γ_0 in subsequent expressions.

With these definitions in hand, we state the main theorem in this paper.

Theorem 1.6 (Continuous Bubbling for NLH)**.** *Let $u(t)$ be a solution of* (1.1) *with initial data $u_0 \in \dot{H}^1$. Let $T_+ = T_+(u_0) \in (0, \infty]$ denote its maximal time of existence and assume that $u(t)$ has finite energy, i.e., $\sup_{t \in [0, T_+)} \bar{E}(u(t)) < \infty$. Then the following hold*

(i) *If $T_+ < \infty$, then there exist a finite energy map $u^* : \mathbb{R}^d \to \mathbb{R}$, an integer $K \ge 1$, and points $\{x^i\}_{i=1}^K \subset \mathbb{R}^d$ such that following holds: let $t_n \to T_+$ be any time sequence. After passing to a subsequence (still denoted by t_n) we can associate to each $i \in \{1, \dots, K\}$ an integer J_i, sequences $a^i_{j,n} \in \mathbb{R}^d$ and $\lambda^i_{j,n} \in (0, \infty)$ for each $j \in \{1, \dots, J_i\}$, with $a^i_{j,n} \to x^i$, $\frac{\lambda^i_{j,n}}{\sqrt{T_+ - t_n}} \to 0$ as $n \to \infty$, and non-zero bubbles $W^i_1, \dots, W^i_{J_i}$ such that*

$$\lim_{n \to \infty} \left(\frac{\lambda^i_{j,n}}{\lambda^i_{k,n}} + \frac{\lambda^i_{k,n}}{\lambda^i_{j,n}} + \frac{|a^i_{j,n} - a^i_{k,n}|}{\lambda^i_{j,n}} \right) = \infty \quad \textit{for all } j \neq k, \tag{1.5}$$

and

$$u(t_n) = u^* + \sum_{i=1}^{K} \sum_{j=1}^{J_i} W^i_j \left(\frac{\cdot - a^i_{j,n}}{\lambda^i_{j,n}} \right) + o_{\dot{H}^1}(1), \tag{1.6}$$

where the error term $o_{\dot{H}^1}(1) \to 0$ *strongly in* $\dot{H}^1$.

(ii) *If* $T_+ = \infty$, *then let* $t_n \to \infty$ *be any time sequence. After passing to a subsequence we can find an integer* $K \geq 0$, *sequences* $a_{j,n} \in \mathbb{R}^d$ *and* $\lambda_{j,n} \in (0, \infty)$ *for each* $j \in \{1, \ldots, K\}$, *with*

$$\lim_{n\to\infty} \frac{|a_{j,n}| + \lambda_{j,n}}{\sqrt{t_n}} = 0 \tag{1.7}$$

and non-zero bubbles $W_1, \ldots, W_K$, *so that*

$$\lim_{n\to\infty} \left(\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} + \frac{|a_{j,n} - a_{k,n}|}{\lambda_{j,n}} \right) = \infty \quad \text{for all } j \neq k,$$

and

$$u(t_n) = \sum_{j=1}^{K} W_j \Big(\frac{\cdot - a_{j,n}}{\lambda_{j,n}} \Big) + o_{\dot{H}^1}(1) \tag{1.8}$$

where the error term $o_{\dot{H}^1}(1) \to 0$ *strongly in* $\dot{H}^1$.

Note that the bubbles obtained in the above decomposition may depend on the sequence of times, which is a similar issue encountered in [JLS25]; however, we can resolve this issue for (1.1) with a very reasonable assumption.

Corollary 1.7. *Let* $u(t)$ *be a solution of* (1.1) *with non-negative initial data* $u_0 \geq 0$ *and* $u_0 \in \dot{H}^1$. *Let* $T_+ = T_+(u_0) \in (0, \infty]$ *denote its maximal time of existence and assume that* $u(t)$ *has finite energy, i.e.,* $\sup_{t\in[0,T_+)} \bar{E}(u(t)) < \infty$. *Then the maps obtained in the decompositions* (1.6) *and* (1.8) *are unique and independent of the sequence of time.*

Theorem 1.6 is a consequence of the following localized bubbling result, in which we denote the ball centered at $x \in \mathbb{R}^d$ with radius $r > 0$ as $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$.

Theorem 1.8 (Localized Bubbling for NLH)**.** *Let* $u(t)$ *be a solution of* (1.1) *with initial data* $u_0 \in \dot{H}^1$. *Let* $T_+ = T_+(u_0) \in (0, \infty]$ *denote its maximal time of existence and assume that* $u(t)$ *has finite energy, i.e.,* $\sup_{t\in[0,T_+)} \bar{E}(u(t)) < \infty$. *Then there exists* $\gamma_0 = \gamma_0(\sup_{t\in[0,T_+)} \bar{E}(u(t))) > 0$ *such that the following holds:*
(i) *If* $T_+ < \infty$, *then for any* $y \in \mathbb{R}^d$,

$$\lim_{t\to T_+} \boldsymbol{\delta}_{\gamma_0}\big(u(t); B(y, \sqrt{T_+ - t})\big) = 0.$$

Moreover, let $t_n \to T_+$ *be any sequence and let* $B(y_n, \rho_n)$ *be any sequence of balls such that* $B(y_n, R_n\rho_n) \subset B(y, \sqrt{T_+ - t})$ *for some sequence* $R_n \to \infty$. *Suppose* α_n, β_n *are sequences with* $\alpha_n \to 0$, $\beta_n \to \infty$, $\lim_{n\to\infty} \beta_n R_n^{-1} = 0$, *and*

$$\lim_{n\to\infty} \bar{E}\big(u(t_n); B(y_n, \beta_n\rho_n) \setminus B(y_n, \alpha_n\rho_n)\big) = 0.$$

Then,

$$\lim_{n\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t_n); B(y_n, \rho_n)\big) = 0.$$

(ii) *If* $T_+ = \infty$, *then for every* $y \in \mathbb{R}^d$,

$$\lim_{t\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t); B(y, \sqrt{t})\big) = 0.$$

Moreover, let $t_n \to \infty$ be any sequence and let $B(y_n, \nu_n)$ any sequence of balls such that $B(y_n, R_n\nu_n) \subset B(y, \sqrt{t_n})$ for some sequence $R_n \to \infty$. Suppose α_n, β_n are sequences with $\alpha_n \to 0$, $\beta_n \to \infty$, $\lim_{n\to\infty} \beta_n R_n^{-1} = 0$ and

$$\lim_{n\to\infty} \bar{E}\big(u(t_n); B(y_n, \beta_n\rho_n) \setminus B(y_n, \alpha_n\rho_n)\big) = 0.$$

Then,

$$\lim_{n\to\infty} \boldsymbol{\delta}_{\gamma_0}\big(u(t_n); B(y_n, \rho_n)\big) = 0.$$

1.3. **Background and Motivation.** A fundamental problem in the analysis of nonlinear partial differential equations (PDEs) is describing the long-time behavior of their solutions. The Soliton Resolution Conjecture asserts that any finite-energy solution to a dispersive PDE asymptotically decomposes into a sum of decoupled solitons that are stationary solutions of the underlying equation, a radiation term that behaves like a solution to the linear flow, and an error term that vanishes in the natural energy norm. This conjecture arose from the numerical experiments of Fermi–Pasta–Ulam–Tsingou [FPU55] and Zabusky–Kruskal [ZK65], which provided evidence that it holds for the Korteweg-de Vries (KdV) equation. Since then, the problem has been extensively studied for the KdV equation as well as for several other integrable models.

Beyond integrable systems, analogues of the Soliton Resolution Conjecture have emerged across various areas of mathematics. In general relativity, the Final State Conjecture (cf. [Kla07]) predicts that generic solutions to Einstein's field equations asymptotically approach a finite number of stationary solutions or Kerr black holes moving apart from each other. In geometric analysis, Soliton Resolution arises naturally in the study of gradient flows associated with conformally invariant variational problems. For example, pioneering works of Struwe [Str85, Str94], Qing [Qin95], Qing–Tian [QT97], and Hong–Tian [HT04] have established Soliton Resolution along a well-chosen sequence of times for the harmonic map and Yang–Mills heat flows.

Motivated by these parabolic works, in this paper we study the energy-critical nonlinear heat flow in dimension $d \geq 3$. Our main result, Theorem 1.6, establishes a continuous-in-time bubble-tree decomposition for all finite-energy solutions of (1.1). More precisely, any solution with uniformly bounded $\dot{H}^1$-norm decomposes into a sum of solitons that may vary along different time sequences, a radiation term that is asymptotically trivial or captured by a weak limit in $\dot{H}^1$, and an error term that vanishes in the energy space. Moreover, when the initial data is non-negative, Corollary 1.7 shows that Theorem 1.6 implies the Soliton Resolution Conjecture, since positive solitons have been classified and are unique up to the symmetries of the equation due to [Oba72, CGS89]. Therefore, Theorem 1.6 extends Struwe's classical compactness result [Str84], which establishes similar decomposition only along a well-chosen sequence of times, while Corollary 1.7 provides the first instance of Soliton Resolution for a non-integrable PDE, beyond radial symmetry, and without restrictions on the size of the initial data.

To explain the significance of our result, we now review some key developments in the literature. In the integrable setting, where tools such as the inverse scattering transform are available, the conjecture is well understood for models including the KdV equation [ES83], the modified KdV equation [Sch06], the one-dimensional cubic nonlinear Schrödinger equation (NLS) [BJM18], the derivative NLS [JLPS19], and, more recently, the Calogero–Moser derivative NLS [KK24].

For non-integrable equations with radial symmetry, where the solitons do not move in space, the conjecture has been settled for the nonlinear wave equation [DKM12, DKM13, DKM23, DKMM22, JK17, CDKM22, JL23b, JL22], damped Klein–Gordon equation [BRS17, GZ23], equivariant self-dual Chern–Simons–Schrödinger equation [KKO22], equivariant harmonic map heat flow [JL23a], and energy-critical nonlinear heat flow [Ary24].

For non-integrable equations, without radial symmetry, Soliton Resolution is known in one dimension for the damped Klein-Gordon equation [CMY21], in the neighborhood of a few solitons

for the energy-critical nonlinear heat flow and the damped Klein-Gordon equation [CMR17a, IN23, Ish25], continuously in time for the harmonic map heat flow [JLS25] or along a sequence of times in dimensions $3 \le d \le 5$ for the energy-critical nonlinear wave equation [DJKM].
In contrast, establishing our main results requires working in any dimension $d \ge 3$, where solitons exhibit only weak decay and no longer enjoy radial symmetry, allowing them to translate in space and potentially behave pathologically (cf. [Din86, DPMPP11, DPMPP13]). Moreover, we impose no restrictions on the size of the initial data, which implies that the nonlinear energy is, in general, non-coercive, unlike the setting of [KM06, GR18]. We overcome these difficulties by introducing new ideas that are robust and adaptable to other nonlinear parabolic flows. In particular, our modified notion of collision intervals, introduced in Section 3, can be used to generalize the results of [JLS25] to higher-dimensional target manifolds.

1.4. **Proof Sketch.** The proofs of the main Theorems 1.6 and 1.8 build on the framework of [JLS25], but require addressing new difficulties that arise in the context of the energy-critical nonlinear heat flow. This includes,

- *Non-coercivity of the energy functional.* The lack of a definite sign for the energy functional (1.2), especially in non-radial settings, prevents the use of standard energy estimates (cf. [Ary24]). To overcome this, we develop new localized energy estimates and use profile decompositions to show that there is no concentration of energy outside the self-similar region, which is a key ingredient in our argument.
- *Absence of energy quantization.* Unlike the case of harmonic maps between the plane and the round two-sphere, solitons for (1.1) do not exhibit quantized energy, thereby preventing a direct application of the collision intervals from [JLS25]. Nevertheless, the existence of a uniform positive lower bound on the energy of any soliton allows us to define suitable collision intervals, which is sufficient to establish our main results.

We first sketch the proof of Theorem 1.8, which in turn is used to prove Theorem 1.6. The argument begins by contradiction. Thus, assume that there is a sequence of times along which the solution deviates from a multi-bubble configuration. Unfortunately, it is difficult to analyze this sequence, and so we give ourselves a bit of room and instead analyze a sequence of time intervals where the solution deviates from a multi-bubble configuration; these sequences of intervals are called collision intervals, for a precise definition, see (3.1).
Thus, consider $[a_n, b_n] \subset [0, T_+)$, a sequence of time intervals where near the endpoints a_n and b_n, $u(t)$ is close to some multi-bubble configuration while inside $[a_n, b_n]$, $u(t)$ deviates away from this multi-bubble configuration. We define K as the smallest integer such that, heuristically, $u(a_n)$ is close to a K-bubble configuration. Note that defining K is straightforward when the energy of each bubble is quantized, as in the case of harmonic maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ since we could simply sum up the energies of each bubble arising in the limit when $n \to \infty$. However, in general, sign-changing stationary solutions could attain a continuum of energies, and thus we need to define K in an approximate sense; see Definition 3.1.
Next, the idea is to use the minimality of K to relate the length of the collision interval to the size of the largest bubble that loses its shape or comes into a collision. In other words, we show that there exist sub-interval $[c_n, d_n] \subset [a_n, b_n]$ and a constant $C_1 > 0$ such that

$$|[c_n, d_n]| \ge C_1 \lambda_{\max,n}^2$$

where $\lambda_{\max,n}^2$ is the largest scale associated with a bubble that comes into a collision. An application of the elliptic bubbling Theorem 2.15 on the interval $[c_n, d_n]$ and a contradiction argument yield a constant $C_2 > 0$ such that

$$\inf_{t \in [c_n, d_n]} \lambda_{\max,n} \|\partial_t u(t)\|_{L^2} \ge C_2.$$

Combining the above two estimates with (1.3) gives

$$\infty > \int_0^{T_+} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \geq \sum_{n\in\mathbb{N}} \int_{c_n}^{d_n} \|\partial_t u(t)\|_{L^2}^2 \,\mathrm{d}t \gtrsim \sum_{n\in\mathbb{N}} 1 = \infty,$$

which is a contradiction, thus completing the proof of Theorem 1.6.

To go from Theorem 1.6 to Theorem 1.8, one key ingredient is to establish no concentration of energy outside the self-similar region. This property is expected to be true in general for a broad class of energy-critical PDEs; however, there are no general techniques to establish such results. We proved this in the radial case [Ary24] using the decay coming from the radial Sobolev embedding; however, new arguments are needed in the non-radial setting. When $T_+ < \infty$, we leverage the L^∞-smoothing estimate to control the nonlinear term to establish that no energy lies outside the ball $B(y, \sqrt{T_+ - t})$. Surprisingly, the case when $T_+ = \infty$ is harder since L^∞ control on a non-compact domain does not yield higher integrability. Here we observe that, given $\phi \in C^\infty$ we have

$$\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \phi^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right) \phi^2 \mathrm{d}x + \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\phi^2 \mathrm{d}x.$$

Therefore, to show that no energy $\dot{H}^1$ energy lives outside the ball $B(y, \sqrt{t})$ for any $y \in \mathbb{R}^d$, it suffices to show that the localized nonlinear energy and the L^{p+1}-norm vanish in this region. By localizing (1.3), it is not difficult to show that the first quantity vanishes. On the other hand, the vanishing of L^{p+1} norm is quite involved and, in particular, relies on a deep result of Ishiwata [Ish18], see Lemma 2.13.

Now let $t_n \to T_+$ be any sequence of times. From Theorem 1.6, we see that $u(t_n)$ approaches K multi-bubble configuration on either $B(y, \sqrt{T_+ - t})$ when $T_+ < \infty$ or $B(y, \sqrt{t})$ when $T_+ = \infty$. In particular, K multi-bubble configurations depend on n. To obtain a finite number of bubbles (independent of n) as in Theorem 1.8 that are asymptotically orthogonal in the sense of (1.5) and (1.7), we apply the Compactness Theorem 2.15 to each bubble obtained in the sequence of multi-bubble configurations arising from Theorem 1.6 and build a new bubble tree configuration by selecting bubbles such that (1.5) and (1.7) are satisfied. The resulting multi-bubble configuration then satisfies all the requirements of Theorem 1.8, thus completing the proof.

1.5. **Notation and Conventions.** We use the following conventions in this paper.

- We denote Strichartz spaces $L_t^p L_x^q$ where the subscripts indicate L^p integral in time and L^q integral in space. In general, we will use Sobolev spaces instead of L^p spaces.
- Some constants that will occur frequently include $p := \frac{d+2}{d-2}$, for $d \geq 3$ and $E_* := \|W\|_{\dot{H}^1}^2$ where W is a non-zero positive stationary solution of (1.4). Furthermore, the inequality $A \lesssim B$ means that $A \leq CB$ for some constant $C > 0$, while $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$.
- An open ball is defined as $B(x, r) = \{z : |z - x| < r\}$ while a parabolic ball $Q_r(x, t) := B(x, r) \times \big(t - r^2, t\big)$ for any $x \in \mathbb{R}^d$, $t > 0$, $r > 0$. For convenience, $Q_1 := B(0, 1) \times (-1, 0)$.
- We will often localize several quantities over the course of this paper. To simplify notation, first, we define the energy densities relevant to the energy-critical heat flow

$$\mathbf{e}(u) := \frac{|\nabla u|^2}{2} - \frac{|u|^{p+1}}{p+1}, \text{ and } \bar{\mathbf{e}}(u) := |\nabla u|^2,$$

  where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Given, $A \subset \mathbb{R}^d$, we measure these quantities localized to this region

$$E(u; A) := \int_A \mathbf{e}(u(t, x))\mathrm{d}x, \text{ and } \bar{E}(u; A) := \int_A \bar{\mathbf{e}}(u(t, x))\mathrm{d}x.$$

Sometimes the domain A might be time-dependent, in which case it is easier to localize using cut-off functions. To that end, given any $\phi \in C^\infty(\mathbb{R}^d)$ we define

$$E_\phi(u) := \int_{\mathbb{R}^d} \mathbf{e}(u(t,x))\phi^2(x)\mathrm{d}x, \text{ and } \bar{E}_\phi(u) := \int_{\mathbb{R}^d} \bar{\mathbf{e}}(u(t,x))\phi^2(x)\mathrm{d}x.$$

- Standard cut-off function will be denoted by $\chi \in C_c^\infty(\mathbb{R}^d)$ where $\chi \equiv 1$ on $B(0,1)$ and $\chi \equiv 0$ outside $B(0,2)$. Rescaling of χ, will be defined as $\chi_R(x) := \chi(x/R)$ for any $R > 0$.

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2. Preliminaries

2.1. **Properties of Stationary Solutions.** In this section, we will recall some standard properties of non-zero solutions to (1.4), show that the Definitions 1.2 of scale and center are well-defined, and establish some natural consequences of these definitions. Let $W : \mathbb{R}^d \to \mathbb{R}$ be a non-zero finite energy solution of (1.4). We will show that the definition of its scale $\lambda(W;\gamma_0)$ and center $a(W;\gamma_0)$ are well-defined.

Lemma 2.1 (Center and scale). *Let $\gamma_0 \in (0, \bar{E}_*/2)$, let $W : \mathbb{R}^d \to \mathbb{R}$ be a non-zero stationary solution, let $\lambda(W) = \lambda(W, \gamma_0)$ be its scale from Definition 1.1 and let $a(W) = a(W, \gamma_0)$ be a choice of center from Definition 1.2. Then $\lambda(W)$ is uniquely defined and strictly positive, and $a(W)$ is well-defined. For all $(b, \mu) \in \mathbb{R}^d \times (0, \infty)$ we have*

$$\lambda\left(W\left(\frac{\cdot - b}{\mu}\right)\right) = \lambda(W)\mu, \text{ and } \left|a\left(W\left(\frac{\cdot - b}{\mu}\right)\right) - b - a(W)\mu\right| \le 2\lambda(W)\mu. \tag{2.1}$$

Proof. Since $\bar{E}(W; B(0,R)) \to \bar{E}(W)$ as $R \to \infty$, it follows that the scale $\lambda(W)$ is well-defined. If $\lambda(W) = 0$, then there exist $a_n \in \mathbb{R}^d$ so that for $n \ge 1$ we have

$$\bar{E}(W; B(a_n, 1/n)) \ge \bar{E}(W) - \gamma_0. \tag{2.2}$$

If $n \ne m$, the $B(a_n, 1/n) \cap B(a_m, 1/m) = \emptyset$. Indeed, otherwise

$$\bar{E}(W) \ge \bar{E}(W; B(a_n, 1/n)) + \bar{E}(W; B(a_m, 1/m)) \ge 2\bar{E}(W) - 2\gamma_0$$

whence $\bar{E}(W) \le 2\gamma_0 < \bar{E}_*$ which contradicts that W is non-zero. Therefore, $\{a_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^d$, and $a_n \to a_\infty$. Passing to the limit in (2.2) gives a contradiction. To see that center $a(W)$ is well-defined, take $\lambda_n \to \lambda(W)$ and $a_n \in \mathbb{R}^d$ such that

$$\bar{E}(W; B(a_n, \lambda_n)) \ge \bar{E}(W) - \gamma_0.$$

As before, we conclude that no two disks $\{B(a_n, \lambda_n)\}_{n=1}^\infty$ can be disjoint. Thus, $a_n \in \mathbb{R}^d$ lie in a compact set and we may assume that $a_n \to a_\infty$ as $n \to \infty$, which is the desired center. We note that $\lambda(W)$ is uniquely defined, but $a(W)$ is defined only up to a distance of $2\lambda(W)$. The properties (2.1) are immediate from the definitions. □

Lemma 2.2 (Decay of stationary solutions). *There exists $\gamma_0 \in (0, \bar{E}_*/2)$ with the following property. For any $0 < \gamma \le \gamma_0$ and any non-zero stationary solution $W : \mathbb{R}^d \to \mathbb{R}$ the exterior energy decays at the following rate:*

$$\bar{E}(W; \mathbb{R}^d \setminus B(a(W;\gamma); R\lambda(W,\gamma))) \le \frac{C}{R^{d-2}}$$

for all $R \ge 1$ with constant $C = C(d, W) > 0$.

Proof. Without loss of generality assume that $a(W;\gamma) = 0$ and $\lambda(W;\gamma) = 1$. Then using Lemma 2.1 in [Pre24] we get precise asymptotic of $|\nabla W|$, which imply the desired estimate since

$$\bar{E}(W;\mathbb{R}^d \setminus B(0;R)) = \int_{B(0,R)^c} |\nabla W|^2 \mathrm{d}x \lesssim \int_R^\infty \frac{r^{d-1}}{(1+r)^{2d-2}} \mathrm{d}r \lesssim \frac{1}{R^{d-2}}.$$

□

Lemma 2.3 (Energy of multi-bubbles)**.** *Let $(y_n, \rho_n, M) \in \mathbb{R}^d \times (0,\infty) \times \mathbb{N}$. Let $\{W_1, \ldots, W_M\}$ be a collection of non-zero stationary solutions, and for each $j \in \{1, \ldots, M\}$ let $(b_{n,j}, \mu_{n,j}) \in B(y_n, \rho_n) \times (0,\infty)$ be sequences such that*

$$\lim_{n\to\infty} \left[\sum_{j\neq k} \left(\frac{\mu_{n,j}}{\mu_{n,k}} + \frac{\mu_{n,k}}{\mu_{n,j}} + \frac{|b_{n,j} - b_{n,k}|}{\mu_{n,j}} \right)^{-1} + \sum_{j=1}^{M} \frac{\mu_{n,j}}{\mathrm{dist}(b_{n,j}, \partial B(y_n, \rho_n))} \right] = 0. \tag{2.3}$$

Then,

$$\lim_{n\to\infty} \bar{E}\Big(\mathbf{W}\big(W_1\big(\frac{\cdot - b_{n,1}}{\mu_{n,1}}\big), \ldots, W_M\big(\frac{\cdot - b_{n,M}}{\mu_{n,M}}\big)\big); B(y_n, \rho_n)\Big) = \sum_{j=1}^{M} \bar{E}(W_j).$$

Proof. To simplify notation within the proof, we use the shorthand $W_{n,j} = W_j\big(\frac{\cdot - b_{n,j}}{\mu_{n,j}}\big)$. Expanding the energy, we obtain

$$\bar{E}(\mathbf{W}(W_{n,1}, \ldots, W_{n,M}); B(y_n, \rho_n)) = \sum_{j=1}^{M} \bar{E}(W_{n,j}; B(y_n, \rho_n)) + 2\sum_{j\neq k} \int_{B(y_n,\rho_n)} (\nabla W_{n,j} \cdot \nabla W_{n,k}) \mathrm{d}x.$$

By the asymptotic orthogonality of the parameters in (2.3), Lemma 2.2 and the invariance of $\dot{H}^1$ norm under translation and rescaling we get

$$\bar{E}(W_{n,j}; B(y_n, \rho_n)) = \bar{E}(W_{n,j}) + o_n(1) = \bar{E}(W_j) + o_n(1)$$

as $n \to \infty$. On the other hand, if $j \neq k$, then

$$\Big| \int_{B(y_n,\rho_n)} (\nabla W_{n,j} \cdot \nabla W_{n,k}) \mathrm{d}x \Big| \leq \int |\nabla W_{n,j}||\nabla W_{n,k}| \mathrm{d}x = o_n(1)$$

by (2.3). Combining the above two displays, we get the desired energy expansion. □

2.2. **Properties of the energy-critical heat flow.** In this section, we will recall the local well-posedness theory for (1.1) and then describe the singular set in the case of finite-time blowup. The following lemma adapts Theorem 1 from [BC96] to our setting,

Lemma 2.4 (Local well-posedness)**.** *Assume $d \geq 3$. Given any $u_0 \in \dot{H}^1$, there exist a time $T_+ = T_+(u_0) > 0$ and a unique function $u \in C([0, T_+], \dot{H}^1)$ with $u(0) = u_0$, which is a classical solution of* (1.1) *on $(0, T_+) \times \mathbb{R}^n$. Moreover, we have,*

(1) *smoothing effect and continuous dependence, namely*

$$\|u(t) - v(t)\|_{\dot{H}^1} + t^{(d-2)/4} \|u(t) - v(t)\|_{L^\infty} \leq C \|u_0 - v_0\|_{\dot{H}^1},$$

for all $t \in (0, T_+)$ where $T_+ = \min\{T_+(u_0), T_+(v_0)\}$ and C can be estimated in terms of $\|u_0\|_{\dot{H}^1}$ and $\|v_0\|_{\dot{H}^1}$.

(2) $\lim_{t\to 0} t^{(n-2)/4} \|u(t)\|_{L^\infty} = 0$.

Remark 2.5. Originally, [BC96] proved local well-posedness in L^q spaces, for solutions to (1.1) with Dirichlet boundary conditions. However, the same argument works on $\mathbb{R}^d$ with $\dot{H}^1$ instead of L^{p+1} space. Observe also that due to the parabolic L^∞-smoothing above and the fact that our main results are about the asymptotic behavior of (1.1) we can assume without loss of generality that the initial data $u_0 \in \dot{H}^1 \cap L^\infty$. See, for instance, Proposition 2.1 in [CMR17b] and the subsequent remark for discussion about the precise gain of regularity for solutions of (1.1).

When $T_+ < \infty$, we will locate the points where energy concentrates. To that end, we first recall a parabolic ε-regularity result established in [GR18] that was proved in dimension $d = 4$, but whose proof is the same in any dimension $d \geq 3$.

Lemma 2.6 (Parabolic ε-regularity)**.** *Given any $k \in \mathbb{N}$, there exist $\varepsilon_0 > 0$ and a constant $C > 0$ such that the following holds. If u is a solution of equation* (1.1) *on Q_1 and satisfies*

$$\varepsilon := \|u\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^4)(Q_1)} < \varepsilon_0$$

then u is smooth on $\overline{Q_{1/2}}$ with bounds

$$\sup_{Q_{1/2}} |D^k u| \leq C\varepsilon.$$

As a consequence, we can define the set of regular and singular when $T_+ < \infty$.

Definition 2.7 (Regular and Singular Points)**.** Let $\mathcal{R} \subset \mathbb{R}^d$ denote the set of regular points, where

$$\mathcal{R} := \left\{ x \in \mathbb{R}^d : \exists r > 0 \text{ such that } \|u\|_{L_t^\infty(\dot{H}_x^1 \cap L_x^4)(Q_r(x,T_+))} < \frac{\varepsilon_0}{2} \right\}$$

where $Q_r(x, T_+) = B(x, r) \times (T_+ - r^2, T_+)$ and $\varepsilon_0 > 0$ is the constant appearing in Lemma 2.6. Let $\mathcal{S} = \mathbb{R}^d \setminus \mathcal{R}$ denote the set of singular points.

Next, we analyze the singular set $\mathcal{S}$.

Theorem 2.8. *Let u be a solution of* (1.1) *such that $T_+ < \infty$ and $\sup_{t \in [0,T_+)} \bar{E}(u(t)) < \infty$. Then the following holds.*

(1) There exist a non-negative integer $L \geq 1$ and a set of points $\{x_1, \ldots, x_L\} \subset \mathbb{R}^d$ such that $\mathcal{S} = \{x_1, \ldots, x_L\}$.
(2) If u^* denote the weak limit of the flow, i.e., $u(t) \rightharpoonup u^* \in \dot{H}^1$ as $t \to T_+$. Then $u(t) \to u^*$ strongly in $\dot{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{S})$ as $t \to T_+$.

Proof. Given any Borel set $A \subset \mathbb{R}^d$ define the measure,

$$\mu(A) := \limsup_{t \to T_+} \int_A (|\nabla u(t,x)|^2 + |u(t,x)|^{p+1}) \mathrm{d}x$$

The finite energy assumption $\sup_{t \in [0,T_+)} \bar{E}(u(t)) < \infty$ implies that $\mu(\mathbb{R}^d) < \infty$. Next, observe that by definition, if $x_0 \in \mathcal{S}$ then for every $r > 0$ there exists $t_r \in (T_+ - r^2, T_+)$ with

$$\int_{B(x_0,r)} (|\nabla u(t_r,x)|^2 + |u(t_r,x)|^{p+1}) \mathrm{d}x \geq \frac{\varepsilon_0^2}{2}.$$

Taking a monotone decreasing sequence $r_n \downarrow 0$ as $n \to \infty$ and the lim sup in the above expression, we see that

$$\mu(\{x_0\}) \geq \lim_{n \to \infty} \mu(B(x_0, r_n)) \geq \frac{\varepsilon_0^2}{2}.$$

Taking any collection of points $\{x_j\}_{j\in\mathcal{S}}$ we see that

$$|\mathcal{S}| \leq \frac{2\mu(\mathbb{R}^d)}{\varepsilon_0^2} < \infty.$$

This proves that $\mathcal{S}$ consists of a finite number of points in $\mathbb{R}^d$. The fact that $L \geq 1$ follows by a contradiction argument; suppose that $L = 0$, then $\mathcal{R} = \mathbb{R}^d$ in which case $u \in C^\infty(\mathbb{R}^d \times [0, T_+])$. In particular, this implies that $\sup_{0\leq t\leq T_+} t^{1/(p-1)}\|u(t)\|_{L^\infty} < \infty$ allowing us to extend the solution $u(t)$ to some larger time $T' > T_+$ contradicting the maximality of $T_+ \in (0, \infty)$. Thus we have proved item (1). For item (2), since $u(t) \in \dot{H}^1$ is a bounded sequence in $\dot{H}^1$, it weakly converges (up to a subsequence) to some $u^* \in \dot{H}^1$ as $t \to T_+$. Let $x \in \mathcal{R}$. Then by definition, there exists $r > 0$ such that $u \in C^k(\overline{Q_{r/2}(x, T_+)})$, for every $k \in \mathbb{N}$. Therefore, for $t \in (T_+ - r^2/4, T_+)$, the sequence of functions $u(t)$ is smooth on the ball $B(x, r/2)$. Applying Arzelà-Ascolli theorem, we see that $u(t) \to u^*$ in $C^k(B(x, r/2))$ as $t \to T_+$ for every $k \in \mathbb{N}$. The proof of item (2) now follows from a standard covering argument and the Sobolev embedding. □

2.3. **Energy Estimates.** In this section, we establish some energy estimates for the energy-critical heat flow (1.1) and use them to propagate smallness of energy for a short-time. Integrals in time and space are with respect to the standard Lebesgue measure, which we omit in the following Lemma for convenience.

Lemma 2.9. *Let $u(t)$ be a solution of* (1.1) *with initial data $u_0 \in \dot{H}^1$. Let $T_+ = T_+(u_0) > 0$ denote its maximal time of existence. Consider $I \subset [0, T_+)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then, for any $t_1, t_2 \in I$ and $t_1 < t_2$ we have*

$$E_\phi(u(t_2)) - E_\phi(u(t_1)) = -\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2 - 2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\nabla u\cdot\nabla\phi)\phi\partial_t u, \tag{2.4}$$

$$\begin{aligned}\bar{E}_\phi(u(t_2)) - \bar{E}_\phi(u(t_1)) = &-2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2 + 2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|u|^{p-1}u(\partial_t u)\phi^2 \\ &- 4\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\nabla u\cdot\nabla\phi)\phi\partial_t u.\end{aligned} \tag{2.5}$$

Furthermore, we have the following estimates,

$$|E_\phi(u(t_2)) - E_\phi(u(t_1))| \leq \int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|\nabla u|^2|\nabla\phi|^2\right)^{1/2}, \tag{2.6}$$

$$\begin{aligned}\left|\bar{E}_\phi(u(t_2)) - \bar{E}_\phi(u(t_1))\right| \leq\ &2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2 + \left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2} \\ &+ 4\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|\nabla u|^2|\nabla\phi|^2\right)^{1/2},\end{aligned} \tag{2.7}$$

$$\bar{E}_\phi(u(t_2)) - \bar{E}_\phi(u(t_1)) \leq 4\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|\nabla u|^2|\nabla\phi|^2 + 2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2}, \tag{2.8}$$

$$\begin{aligned}\bar{E}_\phi(u(t_2)) - \bar{E}_\phi(u(t_1)) \leq\ &2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|u|^{2p}\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2} \\ &+ 4\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\right)^{1/2}\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}|\nabla u|^2|\nabla\phi|^2\right)^{1/2}.\end{aligned} \tag{2.9}$$

.

Proof. The first identity (2.4) follows from

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}(u(t))\phi^2\mathrm{d}x\mathrm{d}t &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\nabla u\cdot\nabla\partial_t u - |u|^{p-1}u\partial_t u)\phi^2\mathrm{d}x\mathrm{d}t\\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\mathrm{div}(\nabla u\partial_t u) - (\Delta u + |u|^{p-1}u)\partial_t u)\phi^2\mathrm{d}x\mathrm{d}t\\ &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\partial_t u)^2\phi^2\mathrm{d}x\mathrm{d}t - 2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}(\nabla u\cdot\nabla\phi)\phi\partial_t u\mathrm{d}x\mathrm{d}t.\end{aligned}$$

The identity (2.5) can be derived similarly. The remaining inequalities (2.6), (2.8), and (2.9) follow by applications of Cauchy-Schwarz and Young's inequality. □

Lemma 2.10 (Short-time propagation of small energy)**.** *Let $u(t)$ be a solution to* (1.1) *with initial data $u(0) = u_0 \in \dot{H}^1$. Let $T_+ = T_+(u_0)$ denote its maximal time of existence and assume that $\sup_{t\in[0,T_+)}\|u(t)\|_{\dot{H}^1} < \infty$. Let $0 < \sigma_n < \tau_n < T_+$ be two sequences of times such that $\sigma_n, \tau_n \to T_+$ as $n \to \infty$ and $\lim_{n\to\infty}(\tau_n - \sigma_n) = 0$. Let W be a stationary solution (possibly zero) and let $r_n > 0$ be a sequence such that $\lim_{n\to\infty}(\tau_n - \sigma_n)r_n^{-2} = 0$. If*

$$\lim_{n\to\infty}\bar{E}(u(\sigma_n) - W; B(0, 2r_n)) = 0,$$

then

$$\lim_{n\to\infty}\bar{E}(u(\tau_n) - W; B(0, r_n)) = 0. \tag{2.10}$$

Next, let $\varepsilon_n > 0$ be a sequence with $\varepsilon_n < r_n$ for all n and such that $\lim_{n\to\infty}(\tau_n - \sigma_n)\varepsilon_n^{-2} = 0$. Let $L \in \mathbb{N}$, $L \geq 1$, $\{x_\ell\}_{\ell=1}^L \subset \mathbb{R}^d$ such that that the balls $B(x_\ell, \varepsilon_n)$ are disjoint and satisfy $B(x_\ell, \varepsilon_n) \subset B(0, r_n)$ for each $n \in \mathbb{N}$ and $\ell \in \{1, \ldots, L\}$. Moreover, $|x_\ell - x_m| \geq 5\varepsilon_n$ when $\ell \neq m$. If

$$\lim_{n\to\infty}\bar{E}(u(\sigma_n) - W; B(0, 2r_n) \setminus \cup_{\ell=1}^L B(x_\ell, \varepsilon_n/2)) = 0,$$

then

$$\lim_{n\to\infty}\bar{E}(u(\tau_n) - W; B(0, r_n) \setminus \cup_{\ell=1}^L B(x_\ell, \varepsilon_n)) = 0. \tag{2.11}$$

Proof. We prove (2.10). Set $v(t) := u(t) - W$. Then,

$$\partial_t v - \Delta v = |u|^{p-1}u - |W|^{p-1}W.$$

Then using the same idea as in (2.8) with a smooth cut-off function $\phi_n \in C_c^\infty(\mathbb{R}^d)$ supported on $B(0, 2r_n)$ we get

$$\begin{aligned}\bar{E}_\phi(v(\tau_n)) \lesssim \bar{E}(v(\sigma_n); B(0, 2r_n)) &+ \frac{(\tau_n - \sigma_n)^{1/2}}{r_n}\left(\int_{\sigma_n}^{\tau_n}\int_{\mathbb{R}^d}|\partial_t u|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &+ \left(\int_{\sigma_n}^{\tau_n}\int_{\mathbb{R}^d}|\partial_t u|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_{\sigma_n}^{\tau_n}\int_{B(0,2r_n)}|u|^{2p}\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\ &+ \left(\int_{\sigma_n}^{\tau_n}\int_{\mathbb{R}^d}|\partial_t u|^2\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_{\sigma_n}^{\tau_n}\int_{B(0,2r_n)}|W|^{2p}\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.\end{aligned} \tag{2.12}$$

By L^∞-smoothing in Lemma 2.4 and using $\lim_{n\to\infty}(\tau_n - \sigma_n) = 0$ we get

$$\int_{\sigma_n}^{\tau_n}\int_{B(0,2r_n)}|u|^{2p}\mathrm{d}x\,\mathrm{d}t \lesssim \frac{\tau_n - \sigma_n}{\sigma_n} \to 0, \text{ as } n \to \infty.$$

Next, using the decay of any stationary solution W from Lemma 2.1 in [Pre24] and $\lim_{n\to\infty}(\tau_n - \sigma_n) = 0$ we get

$$\int_{\sigma_n}^{\tau_n} \int_{B(0,2r_n)} |W|^{2p} \mathrm{d}x \, \mathrm{d}t \lesssim (\tau_n - \sigma_n) \to 0, \text{ as } n \to \infty.$$

Finally, using the energy identity (1.3)

$$\int_{\sigma_n}^{\tau_n} \int_{\mathbb{R}^d} |\partial_t u|^2 \mathrm{d}x \, \mathrm{d}t \leq \int_{\sigma_n}^{T_+} \int_{\mathbb{R}^d} |\partial_t u|^2 \mathrm{d}x \, \mathrm{d}t \to 0, \text{ as } n \to \infty.$$

This shows that all the error terms in (2.12) are asymptotically small, and thus smallness of the energy $\bar{E}(v(\sigma_n); B(0, 2r_n))$ can be transferred to smallness of $\bar{E}(v(\tau_n); B(0, r_n))$ by using the fact that $\phi \equiv 1$ on $B(0, r_n)$.
The proof of (2.11) starts with (2.12) but uses a different cut-off function, which is supported on $B(0, 2r_n) \setminus \cup_{\ell=1}^{L} B(x_\ell, \varepsilon_n/2)$ such that $\phi_n \equiv 1$ on the region $B(0, r_n) \setminus \cup_{\ell=1}^{L} B(x_\ell, \varepsilon_n)$, satisfying the bound $|\nabla \phi_n| \lesssim \varepsilon_n^{-1}$. Then, one can control the error terms following the same reasoning as above. □

2.4. **Concentration properties of the heat flow.** The goal of this section is to establish a crucial fact that energy cannot concentrate outside the self-similar scale, which is expected in type-II blowup scenario. Similar results are known for many other PDEs, for instance, energy-critical nonlinear wave equation [DJKM], wave maps [CTZ93, STZ92], and harmonic map heat flow [JLS25]. Due to the lack of finite speed of propagation, we cannot use the techniques developed for hyperbolic equations, while the lack of a coercive energy for the energy-critical heat flow (1.1) prevents us from using the arguments developed for the harmonic map heat flow when $T_+ = \infty$.

Lemma 2.11 (No self-similar energy concentration in the finite-time blowup case)**.** *Let $u(t)$ be a solution of* (1.1) *with initial data $u_0 \in \dot{H}^1$ such that the maximal time of existence $T_+ = T_+(u_0) < \infty$ and $\sup_{t\in[0,T_+)} \bar{E}(u(t)) < \infty$. Let $x_0 \in \mathcal{S}$ be a singular point as in* (2.7) *and let $r > 0$ be sufficiently small such that $B(x_0, r) \cap (\mathcal{S} \setminus \{x_0\}) = \emptyset$. Then,*

$$\lim_{t\to T_+} \bar{E}(u(t); B(x_0, r) \setminus B(x_0, \alpha\sqrt{T_+ - t})) = \bar{E}(u^*; B(x_0, r)) \tag{2.13}$$

for any $\alpha > 0$. Here, u^* denotes the weak limit of the flow, i.e., $u(t) \rightharpoonup u^*$ as $t \to T_+$. In particular, there exist $T_0 < T_+$ and functions $\nu, \xi : [T_0, T_+) \to (0, \infty)$ such that $\lim_{t\to T_+}(\nu(t) + \xi(t)) = 0$ and the following hold

$$\lim_{t\to T_+} \Big(\frac{\xi(t)}{\sqrt{T_+ - t}} + \frac{\sqrt{T_+ - t}}{\nu(t)} \Big) = 0, \quad \lim_{t\to T_+} \bar{E}(u(t); B(x_0, \nu(t)) \setminus B(x_0, \xi(t))) = 0. \tag{2.14}$$

Proof. The proof method is the same as in [Ary24]. The key point is L^∞-smoothing and the fact that $T_+ < \infty$. Consider a smooth radial cut-off function $\phi \in C_c^\infty(B(x_0, 2r_2) \setminus B(x_0, r_1/2))$ such that $\phi \equiv 1$ on $B(x_0, r_2) \setminus B(x_0, r_1)$ and $\phi \equiv 0$ outside $B(x_0, 2r_2) \setminus B(x_0, r_1/2)$ for any

$0 < r_1 < r_2$. Using (2.5) we see that for each $0 < s < \tau < T_+$ we have,

$$\begin{aligned}
&\left|\bar{E}_\phi(u(\tau)) - \bar{E}_\phi(u(s))\right| \\
&\quad + \left(\int_s^\tau \int_{\mathbb{R}^d} (\partial_t u)^2 \phi^2 |\nabla \phi|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \left(\int_s^\tau \int_{\mathbb{R}^d} |\nabla u|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \\
&\lesssim \int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t + \sqrt{\int_s^\tau \frac{1}{t}\int |u|^{p+1}\mathrm{d}x\mathrm{d}t}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 \mathrm{d}t} \\
&\quad + \frac{(T_+ - s)^{1/2}}{r_1}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 \mathrm{d}t} \\
&\lesssim \int_s^{T_+} \|\partial_t u\|_{L^2}^2 \, \mathrm{d}t + \sqrt{\log(T_+/s)}\sqrt{\int_s^\tau \|\partial_t u\|_{L^2}^2 \mathrm{d}t} \\
&\quad + \frac{(T_+ - s)^{1/2}}{r_1}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t}.
\end{aligned} \tag{2.15}$$

Let $s \to T_+$, then the above estimate implies that $\lim_{s \to T_+} \bar{E}_\phi(u(s))$ exists. Now observe that for some r' such that $0 < r' < \frac{r_1}{2} < r_1$ we have

$$\bar{E}_\phi(u(\tau)) - \bar{E}_\phi(u^*) = \int_{|x-x_0| \geq r'} (\bar{\mathbf{e}}(u) - \bar{\mathbf{e}}(u^*))\phi^2 \mathrm{d}x.$$

Since, $u(t) \to u^*$ strongly in $\dot{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{S})$, the RHS in the above display tends to zero as $\tau \to T_+$. Thus choosing $r_1 = \alpha(T_+ - s)^{1/2}$ and $r_2 = A(T_+ - s)^{1/2}$ in the definition of the cut-off function ϕ, where $0 < \alpha < A$ and sending $\tau \to T_+$ in (2.15) we get

$$\left|\bar{E}_\phi(u^*) - \bar{E}_\phi(u(s))\right| \lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t + \sqrt{\log(T_+/s)}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t} + \frac{1}{\alpha}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t}.$$

Therefore

$$\lim_{s \to T_+} \bar{E}_\phi(u(s)) = 0.$$

Thus, for any $0 < \alpha < A$ we have

$$\lim_{t \to T_+} \bar{E}(u(t); B(x_0, A\sqrt{T_+ - t}) \setminus B(x_0, \alpha\sqrt{T_+ - t})) = 0. \tag{2.16}$$

If instead, we set $r_1 = \alpha(T_+ - s)^{1/2}$ and $r_2 = r$ in the definition of the cut-off function ϕ where $r > 0$ small enough such that $B(x_0, r)$ does not contain any other bubbling point, then we have

$$\left|\bar{E}_\phi(u^*) - \bar{E}_\phi(u(s))\right| \lesssim \int_s^{T_+} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t + \sqrt{\log(T_+/s)}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t} + \frac{1}{\alpha}\sqrt{\int_s^{T_+} \|\partial_t u\|_{L^2}^2 \mathrm{d}t}.$$

Therefore, we have

$$\lim_{s \to T_+} \left|\bar{E}_\phi(u^*) - \bar{E}_\phi(u(s))\right| = 0. \tag{2.17}$$

Denote $A(s) = \{x \in \mathbb{R}^d : \alpha\sqrt{T_+ - s}/2 \le |x - x_0| \le \alpha\sqrt{T_+ - s}\}$ and $A(r) = \{x \in \mathbb{R}^d : r \le |x - x_0| \le 2r\}$ then

$$
\begin{aligned}
&\bar{E}(u(s); B(x_0, r) \setminus B(x_0, \alpha\sqrt{T_+ - s})) - \bar{E}(u^*; B(x_0, r)) \\
&= \bar{E}_\phi(u(s)) - \bar{E}_\phi(u^*) - \int_{\{\phi \neq 1\}} \mathbf{e}(u(s))\phi^2 \mathrm{d}x + \int_{\{\phi \neq 1\}} \mathbf{e}(u^*)\phi^2 \mathrm{d}x - \bar{E}(u^*; B(x_0, \alpha\sqrt{T_+ - s})) \\
&= \bar{E}_\phi(u(s)) - \bar{E}_\phi(u^*) - \int_{A(s)} \mathbf{e}(u(s))\phi^2 \mathrm{d}x - \int_{A(r)} \mathbf{e}(u(s))\phi^2 \mathrm{d}x \\
&\quad + \int_{A(r)} \mathbf{e}(u^*)\phi^2 \mathrm{d}x + \int_{A(s)} \mathbf{e}(u^*)\phi^2 \mathrm{d}x - \bar{E}(u^*; B(x_0, \alpha\sqrt{T_+ - s})),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left|\bar{E}(u(s); B(x_0, r) \setminus B(x_0, \alpha\sqrt{T_+ - s})) - \bar{E}(u^*; B(x_0, r))\right| \\
&\lesssim \left|\bar{E}_\phi(u(s)) - \bar{E}_\phi(u^*)\right| + \bar{E}(u(s); B(x_0, \alpha\sqrt{T_+ - s}) \setminus B(x_0, \alpha\sqrt{T_+ - s}/2)) \\
&\quad + \bar{E}(u^*; B(x_0, \alpha\sqrt{T_+ - s})) + \left|\int_{A(r)} \phi^2(\bar{\mathbf{e}}(u^*) - \bar{\mathbf{e}}(u(s)))\mathrm{d}x\right|.
\end{aligned}
$$

By (2.17), (2.16), and strong convergence of $u(t)$ to u^* in $\dot{H}^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \mathcal{S})$, we see that each term above tends to zero as $s \to T_+$. Thus,

$$
\lim_{s \to T_+} \bar{E}(u(s); B(x_0, r) \setminus B(x_0, \alpha\sqrt{T_+ - s})) = \bar{E}(u^*; B(x_0, r)).
$$

This completes the proof of (2.13). One can easily construct the curves ν and ξ such that the first equation in (2.14) holds. This, along with (2.13), implies the second equation in (2.14). □

Showing the same fact in the global case is significantly more challenging. Unfortunately, we are unable to use energy estimates as in the harmonic map heat flow case in [JLS25] since the Dirichlet energy is not the natural energy associated with (1.1). However, we can deduce nontrivial information if we apply energy estimates to the nonlinear energy.

Lemma 2.12 (Nonlinear energy dissipation in the global case)**.** *Let $u(t)$ be the solution to* (1.1) *with initial data $u_0 \in \dot{H}^1$, $T_+ = T_+(u_0) = \infty$ and finite energy $\sup_{t \ge 0} \bar{E}(u(t)) < \infty$. Then for any $y \in \mathbb{R}^d$ and any $\alpha > 0$ we have*

$$
\lim_{t \to T_+} E_\phi[u[t]] = 0,
$$

where $\phi = 1 - \chi(|x - y|/\alpha\sqrt{t})$ for a smooth cut-off function $\chi \in C_c^\infty(B(0, 2))$.

Proof. Let $\varepsilon > 0$ be small enough. Then we can find $T_0 = T_0(\varepsilon) > 0$ such that,

$$
\Big(\int_{T_0}^\infty \int_0^\infty |\partial_t u|^2 \mathrm{d}x\mathrm{d}t\Big)^{1/2} \le \varepsilon.
$$

Next, choose $T_1 \ge T_0$ so that for all $T \ge T_1$

$$
\bar{E}(u(T_0); \mathbb{R}^d \setminus B(y, \alpha\sqrt{T}/4)) \le \varepsilon.
$$

Fix any such $T \geq T_1$. Let $\phi(x) = 1 - \chi(|x-y|/\alpha\sqrt{T})$ where $\chi \in C_c^\infty(B(0,2))$ is a smooth cut-off function. Then using (2.6)

$$\begin{aligned}
&|E_\phi(u(T)) - E_\phi(u(T_0))| \\
&\leq \int_{T_0}^{T}\int_{\mathbb{R}^d} (\partial_t u)^2\phi^2 \mathrm{d}x + 2\left(\int_{T_0}^{T}\int_{\mathbb{R}^d} |\nabla u|^2|\nabla\phi|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2}\left(\int_{T_0}^{T}\int_{\mathbb{R}^d} \phi^2|\partial_t u|^2 \mathrm{d}x\mathrm{d}t\right)^{1/2} \\
&\leq \varepsilon^2 + \frac{2C\varepsilon}{\alpha} \leq C_1\varepsilon,
\end{aligned}$$

for some constant C_1 that depends on $\alpha > 0$ and $\sup_{t\geq 0} \bar{E}(u(t)) < \infty$. Therefore, we get

$$|E_\phi[u[T]]| \leq |E_\phi[u[T]] - E_\phi[u[T_0]]| + |E_\phi[u[T_0]]| \leq 2C_1\varepsilon,$$

which implies

$$\lim_{T\to T_+} E_\phi[u(T)] = 0,$$

as desired. □

Since the nonlinear energy is not coercive, the above estimate is not very helpful as it does not control the $\dot{H}^1$ norm. However, as explained earlier, due to the following identity

$$\bar{E}_\phi(u) = 2E_\phi(u) + \frac{2}{p+1}\int_{\mathbb{R}^d} |u|^{p+1}\phi^2 \mathrm{d}x, \quad \forall \phi \in C^\infty(\mathbb{R}^d),$$

we observe that if the L^{p+1} norm vanishes outside the region $B(y, \alpha\sqrt{t})$ for any $y \in \mathbb{R}^d$, $\alpha > 0$ then using Lemma 2.12 one can conclude that $\lim_{t\to\infty} \bar{E}(u(t); \mathbb{R}^d \setminus B(y, \alpha\sqrt{t})) = 0$. Thus, we first show the following lemma.

Lemma 2.13 (No self-similar energy concentration in the global case I)**.** *Let $u(t)$ be the solution to* (1.1) *with initial data $u_0 \in \dot{H}^1$, $T_+ = \infty$ and $\sup_{t\geq 0} \bar{E}(u(t)) < \infty$. Then for any $y \in \mathbb{R}^d$ and any $\alpha > 0$ we have*

$$\lim_{t\to T_+} \int_{|x-y|\geq\alpha\sqrt{t}} |u|^{p+1}\mathrm{d}x = 0. \tag{2.18}$$

Proof. Let $\phi(x) = 1 - \chi\left(\frac{|x-y|}{\alpha\sqrt{t}}\right)$ where $\chi \in C_c^\infty(B(0,2))$. We will show that

$$\lim_{t\to T_+} \int_{\mathbb{R}^d} |u|^{p+1}\phi^2 \mathrm{d}x = 0,$$

which will give us the desired result. By translational invariance, assume without loss of generality that $y = 0$. Recall Theorem 1.4 in [Ish18], which states that

$$\lim_{t\to T_+} \mathrm{dist}_{L^{p+1}}(u(t), E_\infty(u_0)) = 0, \tag{2.19}$$

where

$$E_\infty(u_0) = \left\{ \sum_{j=1}^{n} (\lambda^j)^{\frac{(N-2)}{2}} \psi^j\left(\lambda^j(\cdot - x^j)\right) \,\Big|\, n \in \mathbb{N}\cup\{0\},\ \psi^j \text{ solve (1.4)},\ \sum_{j=1}^{n} E(\psi_j) \leq E(u_0)\right\}.$$

As a consequence, if the (2.18) is false, then there exist an initial data $u_0 \in \dot{H}^1$ and $\bar{\alpha} > 0$ such that for some sequence $t_n \to T_+$ we have

$$\int_{|x|\geq\bar{\alpha}\sqrt{t_n}} |u(t_n)|^{p+1}\mathrm{d}x \geq \delta > 0. \tag{2.20}$$

Note that for $\alpha \in (0, \bar{\alpha}]$, $\mathbb{R}^d \setminus B(0, \bar{\alpha}\sqrt{t_n}) \subset \mathbb{R}^d \setminus B(0, \alpha\sqrt{t_n})$ and therefore for any $\alpha \in (0, \bar{\alpha}]$ we have

$$\int_{|x| \geq \alpha\sqrt{t_n}} |u(t_n)|^{p+1} \mathrm{d}x \geq \delta > 0. \tag{2.21}$$

We will show that there exists $\alpha^* \in (0, \bar{\alpha}]$ such that (2.21) implies that the nonlinear energy satisfies

$$\lim_{n \to \infty} E_{\phi_n}[u(t_n)] > 0,$$

where $\phi_n(x) = 1 - \chi(|x|/(\alpha^*\sqrt{t_n}))$ which will contradict Lemma 2.12.
As a starting point, using (2.19) we obtain that (up to a subsequence) and for $n \gg 1$, the following decomposition holds

$$u(t_n) = \sum_{j=1}^{K} (\lambda_n^j)^{\frac{(N-2)}{2}} \psi^j(\lambda_n^j(x - x_n^j)) + r_n,$$

such that

(1) $K \geq 1$ since otherwise $\lim_{n\to\infty} \|r_n\|_{L^{p+1}} = 0$, which would contradict (2.20). Furthermore, $K \in \mathbb{N}$ can be chosen to be independent of n by possibly passing to a subsequence since the sequence has finite energy,
(2) each profile ψ^j is a non-trivial stationary solution,
(3) the parameters are orthogonal in the usual sense

$$\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \frac{|x_n^i - x_n^j|^2}{\lambda_n^j \lambda_n^i} \longrightarrow \infty \quad \text{as } n \to \infty, \ \ i \neq j,$$

and up to a subsequence, we can order the scales $0 < \lambda_n^1 < \lambda_n^2 < \cdots < \lambda_n^K$,
(4) the error satisfies $\|r_n\|_{L^{p+1}} \to 0$, as $n \to \infty$,
(5) and we have the following Pythagorean expansion of various norms,

$$\|\nabla u(t_n)\|_{L^2}^2 = \sum_{j=1}^{K} \|\nabla \psi^j\|_{L^2}^2 + \|\nabla r_n\|_{L^2}^2 + o_n(1), \quad \|u(t_n)\|_{L^{p+1}}^2 = \sum_{j=1}^{K} \|\psi^j\|_{L^{p+1}}^2 + o_n(1).$$

Denote $u_n = u(t_n)$ and $\psi_n^j = (\lambda_n^j)^{\frac{(d-2)}{2}} \psi^j(\lambda_n^j(x - x_n^j))$ for each $j \in \mathcal{J} := \{1, \ldots, K\}$. Consider dividing the index set $\mathcal{J}$ into $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3$, where

$$\mathcal{J}_i := \{j \in \{1, \ldots, K\} : \liminf_{n\to\infty} \lambda_n^j \sqrt{t_n} = L_j \in \mathcal{A}_i\}$$

where $\mathcal{A}_1 = \{0\}, \mathcal{A}_2 \subset (0, \infty)$ and $\mathcal{A}_3 = \{\infty\}$. Consider the cut-off functions

$$\phi_n(x) = 1 - \chi\left(\frac{|x|}{\alpha\sqrt{t_n}}\right), \quad \zeta_n^j(z) = 1 - \chi_n^j(z) = 1 - \chi\left(\frac{|z + c_n^j|}{r_n^j}\right)$$

where $c_n^j = \lambda_n^j x_n^j$, $r_n^j = \alpha \lambda_n^j \sqrt{t_n}$, and $\alpha \in (0, \bar{\alpha}]$ will be fixed later depending on the scales of the profile λ_n^j.
Then, from (2.20) we deduce that there exists at least one bad profile with index $j_0 \in \{1, \ldots, K\}$ such that

$$\int_{|x| \geq \alpha\sqrt{t_n}} |\psi_n^{j_0}|^{p+1} \mathrm{d}x \geq \frac{\delta}{K}. \tag{2.22}$$

Note that $j^0 \in \mathcal{J}_1 \cup \mathcal{J}_2$ since otherwise the integral above would vanish as $n \to \infty$, contradicting (2.22). Expanding the nonlinear energy, we get

$$E_{\phi_n}[u_n] = \sum_{j=1}^{K} E_{\phi_n}[\psi_n^j] + \sum_{j \neq k} A_{j,k,n} + E_{\phi_n}[r_n],$$

where $A_{j,k,n}$ contains all cross–terms between distinct profiles and the remainder r_n, which up to $o_n(1)$ errors are of the form

$$\mathsf{I}_n := \int_{\mathbb{R}^d} (\nabla \psi_n^j \cdot \nabla \psi_n^k) \phi_n^2 \mathrm{d}x, \quad \mathsf{II}_n := \int_{\mathbb{R}^d} (\nabla \psi_n^j \cdot \nabla r_n) \phi_n^2 \mathrm{d}x,$$
$$\mathsf{III}_n := \int_{\mathbb{R}^d} |\psi_n^j|^{p-1} \psi_n^j \psi_n^k \phi_n^2 \mathrm{d}x, \quad \mathsf{IV}_n := \int_{\mathbb{R}^d} |\psi_n^j|^{p-1} \psi_n^j r_n \phi_n^2 \mathrm{d}x$$

for $j, k \in \{1, \ldots, K\}$, $j \neq k$. The terms $\mathsf{I}_n, \mathsf{III}_n$ vanish due to the asymptotic orthogonality of parameters associated to the profiles ψ_n^j and ψ_n^k. Thus, we estimate the remaining terms using integration by parts and Hölder's inequality with $\frac{1}{2} + \frac{1}{n} + \frac{1}{p+1} = 1$ to get

$$\begin{aligned}
|\mathsf{IV}_n| &\leq \|\psi_n^j\|_{L^{p+1}}^p \|r_n\|_{L^{p+1}} \to 0, \\
|\mathsf{II}_n| &= \left| -\int_{\mathbb{R}^d} |\psi_n^j|^{p-1} \psi_n^j r_n \phi_n^2 \mathrm{d}x + 2 \int_{\mathbb{R}^d} (\nabla \psi_n^j \cdot \nabla \phi_n) r_n \phi_n \mathrm{d}x \right| \\
&\lesssim \|\psi_n^j\|_{L^{p+1}}^p \|r_n\|_{L^{p+1}} + \|\nabla \psi_n^j\|_{L^2} \|\nabla \phi_n\|_{L^n} \|r_n\|_{L^{p+1}} \to 0
\end{aligned}$$

as $n \to \infty$. The finiteness of $\|\nabla \phi_n\|_{L^n}$ can be ensured by choosing a log cut-off function, see for instance Lemma 3.8 in [FG20]. By the orthogonality of parameters and the vanishing of the error term in L^{p+1} norm, we have $A_{j,k,n} = o_n(1)$ and $E_{\phi_n}[r_n] \geq -o_n(1)$. Combining with (2.12),

$$\begin{aligned}
E_{\phi_n}[u_n] &\geq \sum_{j=1}^{K} E_{\phi_n}[\psi_n^j] - o_n(1) \\
&\geq \sum_{j \in \mathcal{J}_1} E_{\phi_n}[\psi_n^j] + \sum_{j \in \mathcal{J}_2} E_{\phi_n}[\psi_n^j] + \sum_{j \in \mathcal{J}_3} E_{\phi_n}[\psi_n^j] - o_n(1) \\
&\geq \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 - o_n(1),
\end{aligned}$$

where $\mathrm{E}_i := \sum_{j \in \mathcal{J}_i} E_{\phi_n}[\psi_n^j]$ for $i = 1, 2, 3$. We estimate each term carefully. First we analyze two sub-cases $\mathcal{J}_1 = \mathcal{J}_1^b \cup \mathcal{J}_1^\infty$ where

$$\mathcal{J}_1^b = \{j \in \mathcal{J}_2 : \liminf_{n \to \infty} |c_n^j| < \infty\}, \quad \mathcal{J}_1^\infty = \{j \in \mathcal{J}_2 : \liminf_{n \to \infty} |c_n^j| = \infty\}.$$

Therefore, up to a subsequence when $j \in \mathcal{J}_1^b$, $|c_n^j|$ is bounded while $|c_n^j| \to \infty$ when $j \in \mathcal{J}_1^\infty$. Thus,

$$\begin{aligned}
\mathrm{E}_1 &= \sum_{j \in \mathcal{J}_1} E_{\phi_n}[\psi_n^j] \\
&= \sum_{j \in \mathcal{J}_1^b} \int_{\mathbb{R}^d} \left(\frac{|\nabla \psi_n^j|^2}{2} - \frac{|\psi_n^j|^{p+1}}{p+1} \right) \phi_n^2 \mathrm{d}x + \sum_{j \in \mathcal{J}_1^\infty} \int_{\mathbb{R}^d} \left(\frac{|\nabla \psi_n^j|^2}{2} - \frac{|\psi_n^j|^{p+1}}{p+1} \right) \phi_n^2 \mathrm{d}x \\
&= \sum_{j \in \mathcal{J}_1^b} \int_{\mathbb{R}^d} \left(\frac{|\nabla \psi^j|^2}{2} - \frac{|\psi^j|^{p+1}}{p+1} \right) (\zeta_n^j)^2 \mathrm{d}x + \sum_{j \in \mathcal{J}_1^\infty} \int_{\mathbb{R}^d} \left(\frac{|\nabla \psi^j|^2}{2} - \frac{|\psi^j|^{p+1}}{p+1} \right) (\zeta_n^j)^2 \mathrm{d}x \\
&\geq \frac{|\mathcal{J}_1|}{2d} \bar{E}_* - o_n(1),
\end{aligned}$$

since when $n\to\infty$ by the dominated convergence we have

$$\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi^j|^2}{2}-\frac{|\psi^j|^{p+1}}{p+1}\right)(\zeta_n^j)^2\mathrm{d}z\to E[\psi^j]\geq\frac{\bar{E}_*}{d},\ \forall j\in\mathcal{J}_1^b,$$
$$\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi^j|^2}{2}-\frac{|\psi^j|^{p+1}}{p+1}\right)(\zeta_n^j)^2\mathrm{d}z=E[\psi^j]-\int_{\mathbb{R}^d}\mathbf{e}(\psi^j)\chi_n^j(\chi_n^j-2)\mathrm{d}z\geq\frac{\bar{E}_*}{d}-o_n(1),\ \forall j\in\mathcal{J}_1^\infty.$$

Here we used the fact that

$$j\in\mathcal{J}_1^b\implies\lim_{n\to\infty}\frac{|z+c_n^j|}{r_n^j}=\infty,\ \forall z\in\mathbb{R}^d$$
$$j\in\mathcal{J}_1^\infty\implies\lim_{n\to\infty}\int_{\mathbb{R}^d}|\psi^j|^{p+1}(\chi_n^j)^2\mathrm{d}z\leq\|\psi^j\|_{L^\infty}^{p+1}\lim_{n\to\infty}(r_n^j)^d=0.$$

Next, for profiles in $\mathcal{J}_2$, we analyze two sub-cases $\mathcal{J}_2=\mathcal{J}_2^b\cup\mathcal{J}_2^\infty$ where

$$\mathcal{J}_2^b=\{j\in\mathcal{J}_2:\liminf_{n\to\infty}\frac{|x_n^j|}{\sqrt{t_n}}<\infty\},\quad\mathcal{J}_2^\infty=\{j\in\mathcal{J}_2:\liminf_{n\to\infty}\frac{|x_n^j|}{\sqrt{t_n}}=\infty\}.$$

As a consequence, up to a subsequence, $|c_n^j|$ is bounded when $j\in\mathcal{J}_2^b$. On the other hand, $|c_n^j|=\lambda_n^j\sqrt{t_n}\cdot\frac{|x_n^j|}{\sqrt{t_n}}\to\infty$ as $n\to\infty$ when $j\in\mathcal{J}_2^\infty$. Therefore, we get

$$\begin{aligned}
\mathrm{E}_2&=\sum_{j\in\mathcal{J}_2}E_{\phi_n}[\psi_n^j]\\
&=\sum_{j\in\mathcal{J}_2^b}\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi_n^j|^2}{2}-\frac{|\psi_n^j|^{p+1}}{p+1}\right)\phi_n^2\mathrm{d}x+\sum_{j\in\mathcal{J}_2^\infty}\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi_n^j|^2}{2}-\frac{|\psi_n^j|^{p+1}}{p+1}\right)\phi_n^2\mathrm{d}x\\
&\geq\frac{1}{d}\sum_{\mathcal{J}_2^b}\int_{\mathbb{R}^d}|\nabla\psi_n^j|^2\phi_n^2\mathrm{d}x-\frac{2}{p+1}\sum_{j\in\mathcal{J}_2^b}\int_{\mathbb{R}^d}(\nabla\psi_n^j\cdot\nabla\phi_n)\psi_n^j\phi_n\mathrm{d}x+\frac{|\mathcal{J}_2^\infty|\bar{E}_*}{d}-o_n(1)\\
&\geq\frac{1}{d}\sum_{j\in\mathcal{J}_2^b}\left[\int_{\mathbb{R}^d}|\nabla\psi^j|^2(\zeta_n^j)^2\mathrm{d}z-\frac{C}{\alpha\lambda_n^j\sqrt{t_n}}\int_{r_n^j/2\leq|z+c_n^j|\leq r_n^j}|\nabla\psi^j||\psi_n^j|\mathrm{d}z\right]+\frac{|\mathcal{J}_2^\infty|\bar{E}_*}{d}-o_n(1)\\
&\geq\frac{1}{d}\sum_{j\in\mathcal{J}_2^b}\left[\int_{|z|\geq\alpha L_j}|\nabla\psi^j|^2\mathrm{d}z-C\alpha^{d-1}(L_j)^{d-1}\right]+\frac{|\mathcal{J}_2^\infty|\bar{E}_*}{d}-o_n(1),
\end{aligned}\tag{2.23}$$

where in the second inequality we test (1.4) with $\psi_n^j\phi_n^2$ to get

$$\frac{1}{p+1}\int_{\mathbb{R}^d}|\psi_n^j|^{p+1}\phi_n^2\mathrm{d}x=\frac{1}{p+1}\int_{\mathbb{R}^d}|\nabla\psi_n^j|^2\phi_n^2\mathrm{d}x+\frac{2}{p+1}\int_{\mathbb{R}^d}\phi_n\psi_n^j(\nabla\phi_n\cdot\nabla\psi_n^j)\mathrm{d}x$$

and to estimate the nonlinear energy for $j\in\mathcal{J}_2^\infty$, we use

$$\int_{\mathbb{R}^d}|\psi_n^j|^{p+1}\phi_n^2\mathrm{d}x\leq\int_{|z+c_n^j|\geq r_n^j/2}|\psi^j|^{p+1}\mathrm{d}z\leq\int_{|z|\geq|c_n^j|/2}|\psi^j|^{p+1}\mathrm{d}z\to0,\ \text{as } n\to\infty$$

which implies that

$$\begin{aligned}
\sum_{j\in\mathcal{J}_2^\infty}\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi_n^j|^2}{2}-\frac{|\psi_n^j|^{p+1}}{p+1}\right)\phi_n^2\mathrm{d}x&=\sum_{j\in\mathcal{J}_2^\infty}E[\psi^j]+\int_{\mathbb{R}^d}\mathbf{e}(\psi^j)\chi_n^j(2-\chi_n^j)\mathrm{d}z\\
&\geq\frac{|\mathcal{J}_2^\infty|}{d}\bar{E}_*-o_n(1).
\end{aligned}$$

The final inequality in (2.23) follows by controlling the error term introduced by the cut-off function for profiles with index $j \in \mathcal{J}_2^b$,

$$\begin{aligned}
&\left|\frac{2}{p+1}\int_{\mathbb{R}^d} \phi_n \psi_n^j (\nabla\phi_n \cdot \nabla\psi_n^j)\mathrm{d}x\right| \lesssim \frac{1}{\alpha\lambda_n^j\sqrt{t_n}}\int_{r_n^j/2 \le |z+c_n^j| \le r_n^j} |\nabla\psi^j||\psi^j|\mathrm{d}z \\
&\lesssim (\alpha\lambda_n^j\sqrt{t_n})^{d-1}\|\nabla\psi^j\|_{L^\infty}\|\psi^j\|_{L^\infty} \lesssim (\alpha L_j)^{d-1}\|\nabla\psi^j\|_{L^\infty}\|\psi^j\|_{L^\infty}.
\end{aligned}$$

Recall that for $j \in \mathcal{J}_2^b$, $\lim_{n\to\infty}\lambda_n^j\sqrt{t_n} = L_j$ where $L_j \in (0,\infty)$ (after possibly passing to a subsequence). Denote $L^* = \max_{j\in\mathcal{J}_2^b} L_j$ and $L_* = \min_{j\in\mathcal{J}_2^b} L_j$. Choose $\varepsilon > 0$ small enough such that $\varepsilon^2/L^* \in (0,\bar{\alpha})$ and for each $j \in \mathcal{J}_2^b$ we have

$$\bar{E}(\psi^j; \mathbb{R}^d \setminus B(0,\varepsilon^2)) \ge \frac{\bar{E}_*}{2}.$$

Set $\alpha^* = \varepsilon^2/L^*$ then for each $j \in \mathcal{J}_2^b$ we have

$$\begin{aligned}
\int_{|z|\ge\alpha L_j} |\nabla\psi^j|^2\mathrm{d}z - C(\alpha L_j)^{d-1} &\ge \int_{|z|\ge\alpha L^*} |\nabla\psi^j|^2\mathrm{d}z - C\varepsilon^{2d-2} \\
&\ge \int_{|z|\ge\varepsilon^2} |\nabla\psi^j|^2\mathrm{d}z - C\varepsilon^{2d-2} \\
&\ge \frac{\bar{E}_*}{2} - o_\varepsilon(1) > \frac{\bar{E}_*}{4},
\end{aligned}$$

where $C > 0$ is a constant depending on $\sup_{t\ge0}\bar{E}(u(t)) < \infty$. Therefore, we have

$$\begin{aligned}
\mathrm{E}_2 &\ge \frac{1}{d}\sum_{j\in\mathcal{J}_2}\int_{|z|\ge\alpha L_j} |\nabla\psi^j|^2\mathrm{d}z - C\varepsilon^{2d-2} + \frac{|\mathcal{J}_2^\infty|\bar{E}_*}{d} - o_n(1) \\
&\ge \frac{|\mathcal{J}_2|}{4d}\bar{E}_* - o_n(1).
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{E}_3 &= \sum_{j\in\mathcal{J}_3} E_{\phi_n}[\psi_n^j] \\
&= \sum_{j\in\mathcal{J}_3}\int_{\mathbb{R}^d}\left(\frac{|\nabla\psi_n^j|^2}{2} - \frac{|\psi_n^j|^{p+1}}{p+1}\right)\phi_n^2\mathrm{d}x \\
&= \frac{1}{d}\sum_{j\in\mathcal{J}_3}\int_{\mathbb{R}^d}|\psi_n^j|^{p+1}\phi_n^2\mathrm{d}x - \frac{2}{p+1}\sum_{j\in\mathcal{J}_3}\int_{\mathbb{R}^d}(\nabla\psi_n^j\cdot\nabla\phi_n)\psi_n^j\phi_n\mathrm{d}x \\
&\ge \frac{1}{d}\sum_{j\in\mathcal{J}_3}\int_{\mathbb{R}^d}|\nabla\psi_n^j|^2\phi_n^2\mathrm{d}x - \frac{C}{\lambda_n^j\sqrt{t_n}} \ge -o_n(1).
\end{aligned}$$

Therefore, combining the above estimates, we get

$$\begin{aligned}
E_{\phi_n}[u_n] &\ge \mathrm{E}_1 + \mathrm{E}_2 + \mathrm{E}_3 - o_n(1) \\
&\ge \frac{|\mathcal{J}_1|}{2d}\bar{E}_* + \frac{|\mathcal{J}_2|}{4d}\bar{E}_* - o_n(1) \\
&\ge \frac{|\mathcal{J}_1| + |\mathcal{J}_2|}{4d}\bar{E}_* - o_n(1) \\
&\ge \frac{\bar{E}_*}{4d} - o_n(1)
\end{aligned}$$

since the bad profile has index $j^0 \in \mathcal{J}_1 \cup \mathcal{J}_2$. Therefore

$$E_{\phi_n}[u(t_n)] \geq \frac{\bar{E}_*}{4d} - o_n(1),$$

which implies that

$$\lim_{t \to T_+} E_\phi[u(t)] = \lim_{n \to \infty} E_{\phi_n}[u(t_n)] \geq \frac{\bar{E}_*}{4d} > 0.$$

contradicting Lemma 2.12. ☐

Lemma 2.14 (No self-similar energy concentration in the global case II)**.** *Let $u(t)$ be the solution to* (1.1) *with initial data $u_0 \in \dot{H}^1$, $T_+ = \infty$ and $\sup_{t \geq 0} \bar{E}(u(t)) < \infty$. Then for any $y \in \mathbb{R}^d$ and any $\alpha > 0$ we have*

$$\lim_{t \to \infty} \int_{|x-y| \geq \alpha\sqrt{t}} |\nabla u(t)|^2 \mathrm{d}x = 0.$$

Proof. Let $\phi = 1 - \chi(|x-y|/\alpha\sqrt{t})$, where $\chi \in C_c^\infty(B(0,2))$. Then since

$$\bar{E}_\phi(u) = 2E_\phi(u) + \frac{2}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} \phi^2 \mathrm{d}x,$$

by Lemmas 2.12 and 2.13 we see that

$$\lim_{t \to T_+} \int_{\mathbb{R}^d} |\nabla u|^2 \phi^2 \mathrm{d}x = 0.$$

Therefore

$$\lim_{t \to T_+} \int_{|x-y| \geq \alpha\sqrt{t}} |\nabla u|^2 \mathrm{d}x = 0,$$

as desired. ☐

2.5. **Sequential Compactness.** In this section, we establish an elliptic compactness Theorem for Palais-Smale sequences for critical points associated to the equation (1.4). This result is quite classical with connections to concentration-compactness in analysis [Str84] and the Yamabe problem in differential geometry [BM10].

Theorem 2.15 (Elliptic Bubbling)**.** *Let $u_k : \mathbb{R}^d \to \mathbb{R}$ be a sequence of functions in $\dot{H}^1$ such that*

$$\limsup_{k \to \infty} \int_{\mathbb{R}^d} |\nabla u_k|^2 < \infty, \quad \lim_{k \to \infty} \rho_k \| \Delta u_k + |u_k|^{p-1} u_k \|_{L^2} = 0$$

for some sequence $\rho_k \in (0, \infty)$. Then given any sequence $y_k \in \mathbb{R}^d$, there exist a stationary solution $u_\infty \in \dot{H}^1$ (possibly trivial), an integer $m \in \mathbb{N}$, a constant $C > 0$, a sequence $R_k \to \infty$, a collection of elliptic solutions $W_1, \ldots, W_m$ each equipped with translation parameters $\{x_k^i\}_{i=1}^m \in B(y_k, C\rho_k)$ and scales $\{\lambda_k^i\}_{i=1}^m \in (0, \infty)$ such that

$$\begin{aligned}
&\lim_{k \to \infty} \bar{E}(u_k - u_\infty - \sum_{j=1}^m W_j(\cdot - x_k^j / \lambda_k^j); B(y_k, R_k \rho_k))) \\
&+ \sum_{j \neq j'} \left(\frac{\lambda_k^j}{\lambda_k^{j'}} + \frac{\lambda_k^{j'}}{\lambda_k^j} + \frac{|x_k^{j'} - x_k^j|^2}{\lambda_k^{j'} \lambda_k^j} \right)^{-1} + \sum_{j=1}^m \frac{\lambda_k^j}{\mathrm{dist}(x_k^j, \partial B(y_k, C\rho_k))} = 0.
\end{aligned} \tag{2.24}$$

Denote

$$\bar{\mathcal{S}} = \{x \in \mathbb{R}^d : \liminf_{k\to\infty} \lim_{r\to 0} \int_{B(x,r)\cap B(y_k,C\rho_k)} |u_k|^{p+1}\mathrm{d}x \geq \tilde{\varepsilon}\},$$

for some $\tilde{\varepsilon} = \tilde{\varepsilon}(n)$*. Then* $\bar{\mathcal{S}} = \{x^1, \dots, x^l\}$*, where* $l \leq m$*. Furthermore,*

$$\begin{aligned} &u_k(y_k + \rho_k \cdot) \rightharpoonup u_\infty \text{ weakly in } \dot{H}^1(B(0,C)) \\ &u_k(y_k + \rho_k \cdot) \to u_\infty \text{ strongly in } W^{2,2}_{\mathrm{loc}}(B(0,C) \setminus \bar{\mathcal{S}}). \end{aligned} \tag{2.25}$$

For each $i \in \{1, \dots, m\}$ *there exist a finite set of points* $\bar{\mathcal{S}}_i$*, possibly empty and with* $\mathrm{card}(\bar{\mathcal{S}}_i) \leq m$*, such that*

$$u_k(x^i_k + \lambda^i_k \cdot) \to W_j \text{ strongly in } W^{2,2}_{\mathrm{loc}}(\mathbb{R}^d \setminus \bar{\mathcal{S}}_j). \tag{2.26}$$

Finally, there exists an integer $K \geq 0$ *so that*

$$\lim_{k\to\infty} \bar{E}(u_k; B(y_k, R_k\rho_k)) \in [K\bar{E}_*, (K+1)\bar{E}_*). \tag{2.27}$$

Remark 2.16. The above Theorem 2.15 is similar in spirit to Theorem 1.1 in [Top04] for almost harmonic maps from $\mathbb{S}^2 \to \mathbb{S}^2$. The key difficulty in establishing the above theorem stems from the fact that the natural energy associated with (1.4) does not have a definite sign. Note that, unlike in the harmonic map case, we cannot expect to obtain L^∞ neck-estimates since $W^{2,2}(\mathbb{R}^d)$ does not embed into $C^0(\mathbb{R}^d)$ when $d \geq 4$. Lastly observe that as a consequence of the above theorem, we have $\lim_{n\to\infty} \boldsymbol{\delta}(u_n; B(y_n, \widetilde{\widetilde{R}}_n\rho_n)) = 0$ for any sequence $1 \ll \widetilde{R}_n \ll R_n$.

Proof. See Sections 2 and 3 in [Du13], where this argument has been carried out for $u_k \geq 0$ on bounded domains. However, the same argument can be repeated for sign-changing functions u_k on $\mathbb{R}^d$. The main difference is that the bubbles W_j arising from the blow-up argument are not necessarily positive solutions of (1.4). We briefly sketch the argument for the reader's convenience.

Step 1. Sequential Bubbling. By scaling and translational invariance, we can assume that $\rho_k = 1$ and $y_k = 0$. Denote the set of blowup points for the sequence

$$\mathcal{S} = \{x \in \mathbb{R}^d : \lim_{r\to 0} \liminf_{k\to\infty} \int_{B(x,r)\cap B(0,C)} |u_k|^{p+1}\mathrm{d}x \geq \tilde{\varepsilon}\},$$

where we will fix the constant $C > 0$ later and $\tilde{\varepsilon} = \tilde{\varepsilon}(n) > 0$ is a positive constant that appears in the ε-regularity Theorem 2.1 proved in [Du13], which says that $\int_{B(0,r)} |u|^{p+1} \leq \widetilde{\varepsilon(n)}$ implies that $\int_{B(0,\delta)} |\nabla u|^2 \leq C_0$ for small $\delta \in (0,1)$ and some constant $C_0 > 0$. Since $\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} |\nabla u_k|^2 < \infty$, by choosing choose $r > 0$ small enough such that $B(x^i, r) \cap B(x^j, r) = \emptyset$ for $i \neq j$, a standard covering argument implies that $\mathcal{S} = \{x^1, \dots, x^N\}$ for some finite $N \in \mathbb{N}$, and $x^i \in \mathbb{R}^d$ for $1 \leq i \leq N$. We can choose C in the definition of $\mathcal{S}$ small enough such that $\mathcal{S}$ consists of singleton, i.e., $\mathcal{S} = \{x^1\}$.

Step 1.1. Extracting the first bubble. Fix $x \in \overline{B(x^1, r) \cap B(0,C)}$ and let $r_k := r_k(x)$ be the unique radius depending on x such that

$$\int_{B(x,r_k)\cap B(0,C)} |u_k|^{p+1}\mathrm{d}x = \frac{\widetilde{\varepsilon}}{2}.$$

Let $x^1_k \in \overline{B(x^1, r) \cap B(0,C)}$ be the point where $r_k(x)$ attains its minimum. Then define $\lambda^1_k = r_k(x^1_k)$. Thus we have a blowup sequence, $\lambda^1_k \to 0$ and $x^1_k \to x^1$ as $k \to \infty$ such that

$$\int_{B(x^1_k,\lambda^1_k)} |u_k|^{p+1}\mathrm{d}x = \frac{\tilde{\varepsilon}}{2}.$$

Re-scaling the function u_k,

$$\tilde{u}_k(x) = (\lambda_k^1)^{2/(p-1)} u_k(\lambda_k^1 x + x_k^1)$$

and using the ε-regularity proved in Theorem 2.1 in [Du13] we see that since

$$\Delta \tilde{u}_k + |\tilde{u}_k|^{p-1}\tilde{u}_k = (\lambda_k^1)^{\frac{2}{p-1}}(\Delta u_k + |u_k|^{p-1}u_k),$$

the sequence $\tilde{u}_k \to W_1$ in $H^1_{\mathrm{loc}}(\mathbb{R}^d)$ where W_1 solves (1.4) either on $\mathbb{R}^d$ or $\mathbb{R}^d_+$ depending on whether x_k^1 lies in the interior of the domain $\overline{B(x^1, r) \cap B(0, C)}$ or on its boundary. The latter can be ruled out by showing

$$\frac{\lambda_k^1}{\mathrm{dist}(x_k^1, \partial B(0, C))} \to 0, \ k \to \infty$$

which can be done by a contradiction argument, that involves assuming $\frac{\lambda_k^1}{\mathrm{dist}(x_k^1, \partial B(0,C))} \to c \in (0, \infty], \ k \to \infty$ and showing that this gives rise to a solution of (1.4) on the half-space which is known to be trivial by Pohozaev's identity. For more details, see page. 162, Section 3 in [Du13] or the proof of Proposition 2.1 in [Str84].

Step 1.2. Consider the re-normalized sequence

$$v_k(x) = u_k(x) - W_1\left(\frac{\cdot - x_k^1}{\lambda_k^1}\right).$$

If v_k converges (up to subsequence) strongly to u_∞ in $\dot{H}^1(B(x_1, r) \cap B(0, C))$ then we are done. Otherwise, as in Step 1.1, we can find scales $\lambda_k^2 \to 0$ and centers $x_k^2 \to x^1$ such that

$$\int_{B(x_k^2, \lambda_k^2)} |v_k|^{p+1} = \frac{\tilde{\varepsilon}_1}{2} \tag{2.28}$$

for some constant $0 < \tilde{\varepsilon}_1 \leq \tilde{\varepsilon}$. We first claim that

$$\frac{\lambda_k^2}{\lambda_k^1} + \frac{|x_k^1 - x_k^2|}{\lambda_k^1 + \lambda_k^2} \to \infty, \text{ as } k \to \infty,$$

since otherwise there exists some constant $M > 0$ such that

$$\frac{\lambda_k^2}{\lambda_k^1} + \frac{|x_k^1 - x_k^2|}{\lambda_k^1 + \lambda_k^2} \leq M, \qquad \text{as } k \to \infty.$$

This, in turn, would imply that

$$\int_{B(x_k^2, \lambda_k^2)} |v_k|^{p+1} \mathrm{d}x \leq \int_{B((x_k^2 - x_k^1)/\lambda_k^1, \lambda_k^2/\lambda_k^1)} |\tilde{u}_k - W_1|^{p+1} \mathrm{d}x \leq \int_{B(0,M)} |\tilde{u}_k - W_1|^{p+1} \mathrm{d}x \to 0$$

as $k \to \infty$ which contradicts the energy concentration in (2.28). The next subtle point here is to show that no energy is lost between the *neck*-region connecting the new bubble W_2 and the previous bubble W_1. This has been done in Section 4 [Du13] and therefore at the end of this step, we get

$$u_k - (\lambda_k^1)^{\frac{-2}{p-1}} W_1\left(\frac{x - x_k^1}{\lambda_k^1}\right) - (\lambda_k^2)^{\frac{-2}{p-1}} W_2\left(\frac{x - x_k^2}{\lambda_k^2}\right) \to 0$$

strongly in $\dot{H}^1(B(x^1, L\lambda_k^1) \cap B(x^1, L\lambda_k^2))$ for any $L > 0$.

Step 1.3. Iterate and conclude. One can then iterate this process finitely many times to extract the bubble tree as desired with asymptotically orthogonal parameters as in the second display in (2.24).

Step 2. Convergence results. The existence of the weak limit in (2.25) follows from the fact that u_k is a bounded sequence of $\dot{H}^1$ functions. The strong convergence in $W^{2,2}_{\text{loc}}$ away from the blowup points follows from the ε-regularity result from Theorem 2.1 in [Du13]. Thus u_∞ is a smooth stationary solution of (1.4) away from a finite set of points. Then the standard removable singularity theorem, see for instance [CGS89, Lemma 2.1], implies that u_∞ is a smooth solution of (1.4) on $\mathbb{R}^d$. The strong convergence in (2.26) follows from the definition of the blow-up parameters (x_k^i, λ_k^i) and ε-regularity from Theorem 2.1 in [Du13].

Step 3. Energy almost-quantization. The bubble tree convergence and the *no-neck* property established in Section 4 of [Du13] imply the energy identity

$$\lim_{k\to\infty} \bar{E}(u_k; B(0, R_k\rho_k)) = \sum_{j=1}^{m} \bar{E}(W_j).$$

Since $E[W_j] \geq \bar{E}_*$ and we know that

$$\lim_{k\to\infty} \bar{E}(u_k; B(0, R_k\rho_k)) \geq m\bar{E}_*.$$

Furthermore $\sum_{j=1}^m \bar{E}(W_j) \leq C_1$ since the sequence u_k has finite energy. Therefore, we can find an integer between m and an integer less than or equal to $C_1/\bar{E}_*$ such that, up to passing to a subsequence, there exists a non-negative integer $K \geq m$ satisfying

$$\lim_{k\to\infty} \bar{E}(u_k; B(0, R_k)) \in [K\bar{E}_*, (K+1)\bar{E}_*)$$

as desired. □

3. Analysis of Collision Intervals

For convenience, in this section, we let $u(t)$ be a solution of (1.1), with initial data $u_0 \in \dot{H}^1$ defined on the maximal time interval $I_+ = [0, T_+)$ where $T_+ < \infty$ in the finite time blow-up case and $T_+ = \infty$ in the global case. We will also assume that $C' = \sup_{t\in[0,T_+)} \bar{E}(u(t)) < \infty$. Let $0 < \gamma_0 \ll 1$ (in particular $\gamma_0 \ll 1/C'$) such that Lemma 2.2 holds. We fix this choice of γ_0 and drop the subscript γ_0 from $\mathbf{d}_{\gamma_0}$ and $\boldsymbol{\delta}_{\gamma_0}$ and from the notation for scale and center of a stationary solution W, in particular $\lambda(W) = \lambda(W; \gamma_0)$ and $a(W) = a(W; \gamma_0)$. Our goal in this section is to introduce the notion of collision intervals and show that if Theorem 1.6 fails, then these intervals have a nontrivial length.

Definition 3.1 (Collision Interval)**.** Let $K \in \mathbb{N}$ be the smallest number with the following properties. There exist sequences of centers and scales $(y_n, \rho_n, \varepsilon_n) \in \mathbb{R}^d \times (0,\infty)^2$, sequences of times $\sigma_n, \tau_n \in (0, T_+)$ and small (but fixed) $\eta > 0$, satisfying $\varepsilon_n \to 0$, $0 < \sigma_n < \tau_n < T_+$, $\sigma_n, \tau_n \to T_+$, such that

(1) $\boldsymbol{\delta}(u(\sigma_n); B(y_n, \rho_n)) \leq \varepsilon_n$;
(2) $\boldsymbol{\delta}(u(\tau_n); B(y_n, \rho_n)) \geq \eta$;
(3) the interval $I_n := [\sigma_n, \tau_n]$ satisfies $|I_n| \leq \varepsilon_n \rho_n^2$;
(4) $\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n, \rho_n)) \in [K\bar{E}_*, (K+1)\bar{E}_*)$.

Then intervals $[\sigma_n, \tau_n]$ are called collision intervals associated to the energy level K and the parameters $(y_n, \rho_n, \varepsilon_n, \eta)$. We can conveniently package this information in the following notation $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \varepsilon_n, \eta)$.

Remark 3.2. By Definition 1.4 and item (1) in Definition 3.1, we can associate to each sequence of collision intervals $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \varepsilon_n, \eta)$ a sequence $(\xi_n, \nu_n) \in (0,\infty)^2$ with $\lim_{n\to\infty}\left(\frac{\xi_n}{\rho_n} + \frac{\rho_n}{\nu_n}\right) = 0$ such that

$$\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n, 2\nu_n) \setminus B(y_n, 2^{-1}\xi_n)) = 0. \tag{3.1}$$

Using item (3) in Definition 3.1 also allows to assume that

$$|I_n| = \tau_n - \sigma_n \ll \xi_n^2. \tag{3.2}$$

Using Lemma 2.10 with (3.1) and (3.2), we get

$$\lim_{n\to\infty} \sup_{t\in[\sigma_n,\tau_n]} \bar{E}(u(t); B(y_n,\nu_n)\setminus B(y_n,\xi_n)) = 0. \tag{3.3}$$

The same argument works if we either enlarge ξ_n or shrink ν_n in the sense that we can replace (ξ_n,ν_n) by $(\widetilde{\xi}_n,\widetilde{\nu}_n)$ where $\xi_n \ll \widetilde{\xi}_n \ll \rho_n \ll \widetilde{\nu}_n \ll \nu_n$.

Lemma 3.3 (Existence of $K \geq 1$)**.** *If Theorem 1.6 is false, then K is well-defined with $K \geq 1$.*

Proof. Assume that Theorem 1.6 is false. Then there exist $\eta > 0$, $\rho_n \in (0,\infty)$ where $\rho_n \leq \sqrt{T_+ - t_n}$ when $T_+ < \infty$ and $\rho_n \leq \sqrt{t_n}$ when $T_+ = \infty$ and sequences $\alpha_n \to 0$ and $\beta_n \to \infty$ such that for all $n \in \mathbb{N}$ we have

$$\boldsymbol{\delta}(u(\tau_n); B(y_n,\rho_n)) \geq \eta, \ \lim_{n\to\infty} \bar{E}(u(\tau_n); B(y_n,\beta_n\rho_n)\setminus B(y_n,\alpha_n\rho_n)) = 0.$$

The existence of the sequences α_n and β_n follows from Lemma 2.11 or Lemma 2.14 when $\rho_n \simeq \sqrt{T_+ - \tau_n}$ or $\rho_n \simeq \sqrt{\tau_n}$.
Next, we can find sequences σ_n and τ_n such that $\sigma_n < \tau_n$, $\sigma_n, \tau_n \to T_+$, $|[\sigma_n,\tau_n]| \ll \rho_n^2$ and

$$\lim_{n\to\infty} \rho_n^2 \|\partial_t u(\sigma_n)\|_{L^2}^2 = 0.$$

To see this, assume to the contrary. Then there exist constants $c, c_0 > 0$ such that up to a subsequence we have

$$\rho_n^2 \|\partial_t u(t)\|_{L^2}^2 \geq c_0,$$

for all $t \in [\tau_n - c\rho_n^2, \tau_n]$. However, this yields a contradiction since $u(t)$ has finite energy and therefore by the energy identity (1.3) we have

$$\infty > \int_0^{T_+} \int_{\mathbb{R}^d} |\partial_t u(t)|^2 \mathrm{d}x \ \mathrm{d}t \geq \sum_n \int_{\tau_n - c\rho_n^2}^{\tau_n} \int_{\mathbb{R}^d} |\partial_t u(t)|^2 \mathrm{d}x \ \mathrm{d}t \geq c_0 \sum_n \int_{\tau_n - c\rho_n^2}^{\tau_n} \rho_n^{-2} \, \mathrm{d}t = \infty.$$

Using (2.7), with $t_1 = \sigma_n$, $t_2 = \tau_n$, cut-off function $\phi \in C_c^\infty(B(y_n,\beta_n\rho_n)\setminus B(y_n,\alpha_n\rho_n))$ and showing that the error terms vanish as in the proof of Lemma 2.10, we get

$$\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n, 2^{-1}\beta_n\rho_n)\setminus B(y_n, 2\alpha_n\rho_n)) = 0. \tag{3.4}$$

Applying the sequential bubbling Theorem 2.15 to $u(\sigma_n)$, we obtain a bubble tree decomposition (2.24) along some subsequence of σ_n and for some sequence $R_n \to \infty$. Since energy vanishes in the neck region (3.4), we see that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(\sigma_n); B(y_n,\rho_n)) = 0.$$

By Lemma 2.3 we can find an integer $K \geq 0$ such that

$$\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n,\rho_n)) \in [K\bar{E}_*, (K+1)\bar{E}_*).$$

Thus, we have verified all the items in the Definition 3.1 for the interval $[\sigma_n,\tau_n]$, which shows that K is well defined and that $K \geq 0$.
To see that $K \geq 1$, we argue by contradiction. Suppose $K = 0$. Let ξ_n, ν_n be sequences as in Remark 3.2. Then, since $K = 0$, we get that $\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n,\rho_n)) < \bar{E}_*$. This implies that $u(\sigma_n)$ cannot be a close to any multi-bubble configuration and therefore by item (1) in

Definition 3.1, we get that $\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n, \rho_n)) = 0$. Using Lemma 2.10 and (3.4) we get that

$$\bar{E}(u(\tau_n); B(y_n, \rho_n)) = o_n(1),$$

which contradicts item (2) in Definition 3.1. Thus $K \geq 1$. □

For the remainder of this section, assume that Theorem 1.6 is false. We will show that this implies a nontrivial lower bound on the length of the collision intervals. Let $K \geq 1$ be as in Lemma 3.3 and $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \varepsilon_n, \eta)$, where

$$y_n \in \mathbb{R}^d, \rho_n \in (0, \infty), \varepsilon_n \to 0, \eta > 0, 0 < \sigma_n < \tau_n < T_+, \sigma_n \to T_+, \tau_n \to T_+$$

are parameters that satisfy the requirements of Definition 3.1. We first prove a very general lower bound on the size of the intervals where the solution is initially close and later far from a multi-bubble configuration. We will call these *bad* intervals.

Lemma 3.4 (Lower bound on the length of *bad* intervals). *There exists $\eta_0 > 0$ such that for all $\eta \in (0, \eta_0)$, there exist constants $\varepsilon, c_0 > 0$ such the following holds; let $[\sigma, \tau] \subset [\sigma_n, \tau_n]$ be any subset such that*

$$\boldsymbol{\delta}(u(\sigma); B(y_n, \rho_n)) \leq \varepsilon, \quad \boldsymbol{\delta}(u(\tau); B(y_n, \rho_n)) \geq \eta,$$

let $\vec{W} = (W_1, \ldots, W_M)$ be any collection of non-constant stationary solutions, $\vec{\nu} = (\nu, \nu_1, \ldots, \nu_M) \in (0, \infty)^{M+1}$, $\vec{\xi} = (\xi, \xi_1, \ldots, \xi_M) \in (0, \infty)^{M+1}$ any admissible vectors in the sense of Definition 1.4 such that,

$$\varepsilon \leq \mathbf{d}(u(\sigma), \mathbf{W}(\vec{W}); B(y_n, \rho_n); \vec{\nu}, \vec{\xi}) \leq 2\varepsilon.$$

Then

$$\tau - \sigma \geq c_0 \max_{j \in \{1, \ldots, M\}} \lambda(W_j)^2.$$

Remark 3.5 (Proof Sketch). Since the proof of Lemma 3.4 is quite involved, we give a summary of the key ideas. As usual, we will argue by contradiction. Thus, there exists a sequence of intervals $[s_n, t_n] \subset [\sigma_n, \tau_n]$ such that $|[s_n, t_n]| \ll \lambda_{\max,n}^2$. The idea then is to contradict the minimality of $K \geq 1$ since the interval size of the $[s_n, t_n]$ is too short compared to the scale $\lambda_{\max,n}$ implying that the collisions are captured on small balls $B(y_n', \rho_n') \subset B(y_n, \rho_n)$ with $\rho_n' \ll \rho_n$. As we do not see the large scales $\lambda_{\max,n}$ in these small balls $B(y_n', \rho_n')$, we deduce that these small balls must carry strictly smaller energy in the sense of the last item in Definition 3.1, which will contradict the minimality of K.
To make the above argument precise, it will be helpful to organize the bubbles that will arise when the localized distance $\mathbf{d}$ vanishes. To that end, we first distinguish the bubbles based on the size of their $\dot{H}^1$-interaction. In particular, if this interaction vanishes, then we say that the bubbles are asymptotically orthogonal.

Definition 3.6 (Asymptotic Orthogonality of Scales). We say that two triples $(W_j, a_{j,n}, \lambda_{j,n})$ and $(W_{j'}, a_{j',n}, \lambda_{j',n})$ are *asymptotically orthogonal* if

$$\lim_{n\to\infty} \Big(\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|a_{j,n} - a_{j',n}|^2}{\lambda_{j,n}\lambda_{j',n}} \Big) = \infty, \tag{3.5}$$

where $W_j, W_{j'}$ are non-zero stationary solutions of (1.4), $a_{j,n}, a_{j',n} \in \mathbb{R}^d$ are sequences of points, and $\lambda_{j,n}, \lambda_{j',n} \in (0, \infty)$ are sequences of scales. We will use the short hand $(W_j, a_{j,n}, \lambda_{j,n}) \perp (W_{j'}, a_{j',n}, \lambda_{j',n})$ if the two triples are asymptotically orthogonal. See Proposition B.2 in [FG20] to understand the connection between (3.5) and the integral interaction between the bubbles W_j and $W_{j'}$ in the case when $W_j, W_{j'} \geq 0$.

Using the above notion of asymptotic orthogonality, we can organize a family of bubbles into a tree-like structure.

Definition 3.7 (Bubble Tree)**.** Given two collections of stationary solutions $\mathfrak{h}_1 = \{W_n\}_{n=1}^{\infty}$ and $\mathfrak{h}_2 = \{\widetilde{W}_n\}_{n=1}^{\infty}$, then $\mathfrak{h}_1 \prec \mathfrak{h}_2$ iff

$$\frac{\lambda(\widetilde{W}_n)}{\lambda(W_n)} \to \infty \text{ and } \exists C > 0 \text{ such that } B(a(W_n), \lambda(W_n)) \subset B(a(\widetilde{W}_n), C\lambda(\widetilde{W}_n)) \text{ for all } n \gg 1.$$

Then we say that $\mathfrak{h}_1$ is the parent and $\mathfrak{h}_2$ is its child. We will also allow for equality in the above relation by using the notation $\mathfrak{h}_1 \preceq \mathfrak{h}_2$. Given $M \in \mathbb{N}$ consider the collection $\{\mathfrak{h}_1, \dots, \mathfrak{h}_M\}$ where $\mathfrak{h}_i = \{W_{k,i}\}_{k=1}^{\infty}$ and $W_{k,i}$ are stationary solutions. We define a *root* element $\mathfrak{h}_j$ as an element that is not a child of any parent $\mathfrak{h}_{j'}$ for $j' \in \{1, \dots, M\}$. We define the collection of all root-indices as

$$\mathcal{R} := \{j \in \{1, \dots, M\} \mid \mathfrak{h}_j \text{ is a root }\}.$$

Finally, to each root $\mathfrak{h}_j$ we can define the *bubble tree* as the following collection $\mathcal{T}(j) := \{\mathfrak{h}_{j'} \mid \mathfrak{h}_{j'} \preceq \mathfrak{h}_j\}$ and $\mathcal{D}(j)$ as the set of all maximal elements (with respect to the partial order $\preceq$) of the pruned tree $\mathcal{T}(j) \setminus \{\mathfrak{h}_j\}$.

Proof of Lemma 3.4. Assume that Lemma 3.4 does not hold. Then there exist a sequence of intervals $[s_n, t_n] \subset [\sigma_n, \tau_n]$ such that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(s_n); B(y_n, \rho_n)) = 0, \quad \lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(y_n, \rho_n)) > 0, \tag{3.6}$$

a sequence of integers $M_n \geq 0$, sequences of M_n-bubble configurations $\mathbf{W}(\vec{W}_n)$, where $\vec{W}_n = (W_{1,n}, \dots, W_{M_n,n})$ and sequences of vectors $\vec{\nu}_n = (\nu_n, \nu_{1,n}, \dots, \nu_{M_n,n}) \in (0,\infty)^{M_n+1}, \vec{\xi}_n = (\xi_n, \xi_{1,n}, \dots, \xi_{M_n,n}) \in (0,\infty)^{M_n+1}$ such that

$$\lim_{n\to\infty} \mathbf{d}(u(s_n), \mathbf{W}(\vec{W}_n); B(y_n, \rho_n); \vec{\nu}_n, \vec{\xi}_n) = 0, \tag{3.7}$$

and the largest scale $\lambda_{\max,n} := \max_{j=1,\dots,M_n} \lambda(W_{j,n})$ satisfies $(t_n - s_n)^{1/2} \ll \lambda_{\max,n}$.
We can assume that $M_n = M$ is a fixed integer by possibly passing to a subsequence. Consider the collection $\{\mathfrak{h}_1, \dots, \mathfrak{h}_M\}$ where $\mathfrak{h}_j = \{W_{j,n}\}_{n=1}^{\infty}$ for $j \in \{1, \dots, M\}$. Then construct a bubble tree as in Definition 3.7. By definition, for any $j, j' \in \mathcal{R}$ we can find a sequence $\widetilde{R}_n \to \infty$ such up to a subsequence we have

$$B(a(W_{j,n}), 4R_n\lambda(W_{j,n})) \cap B(a(W_{j',n}), 4R_n\lambda(W_{j',n})) = \emptyset$$

for any sequence $R_n \leq \widetilde{R}_n$, where recall that $a(W_{j,n}), \lambda(W_{j,n})$ denote the center and the scale of the stationary solution $W_{j,n}$. Then the decay estimate 2.2 implies that for any $j \in \mathcal{R}$ and any sequence $R_n \to \infty$ we have

$$\lim_{n\to\infty} \bar{E}(W_{j,n}; \mathbb{R}^d \setminus B(a(W_{j,n}); 4^{-1}R_n\lambda(W_{j,n}))) = 0,$$

which in turn combined with (3.7) yields

$$\lim_{n\to\infty} \bar{E}(u(s_n); B(y_n, \rho_n) \setminus \cup_{j\in\mathcal{R}} B(a(W_{j,n}), 4^{-1}R_n\lambda(W_{j,n}))) = 0. \tag{3.8}$$

Next, applying Theorem 2.15 to the sequence of stationary solutions $W_{j,n}$ and passing to a joint subsequence, we find a sequence $M_j \geq 0$ of non-negative integers, a sequence $\breve{R}_n \leq \widetilde{R}_n$ with $1 \ll \breve{R}_n \ll \xi_n\lambda_{\max,n}^{-1}$, stationary solutions $\mathcal{W}_{j,0}$, non-zero stationary solutions $\mathcal{W}_{j,k}$, scales

$\Lambda_{j,k,n} \ll \lambda(W_{j,n})$ and points $p_{j,k,n} \in B(a(W_{j,n}), C\lambda(W_{j,n}))$ for each j and $k \in \{1, \ldots, M_j\}$, satisfying (2.25), (2.26), and

$$\begin{aligned} &\lim_{n\to\infty} \bar{E}(W_{j,n} - \mathcal{W}_{j,0}\Big(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})}\Big) - \sum_{k=1}^{M_j} \mathcal{W}_{j,k}\Big(\frac{\cdot - p_{j,k,n}}{\Lambda_{j,k,n}}\Big); B(a(W_{j,n}), 4R_n\lambda(W_{j,n}))) \\ &+ \sum_{k\neq k'} \Big(\frac{\Lambda_{j,k,n}}{\Lambda_{j,k',n}} + \frac{\Lambda_{j,k',n}}{\Lambda_{j,k,n}} + \frac{|p_{j,k,n} - p_{j,k',n}|^2}{\Lambda_{j,k,n}\Lambda_{j,k',n}}\Big)^{-1} + \sum_{k=1}^{M_j} \frac{\Lambda_{j,k,n}}{\operatorname{dist}(p_{j,k,n}, \partial B(a(W_{j,n}), C\lambda(W_{j,n}))} = 0. \end{aligned} \tag{3.9}$$

Here $C > 0$ is some finite constant, and R_n is a sequence, to be fixed below, such that $1 \ll R_n \leq \breve{R}_n$. To differentiate the weak limits $\mathcal{W}_{j,0}$ (which could be trivial) with the stationary solutions $\mathcal{W}_{j,k}$ we will call $\mathcal{W}_{j,0}$ as body maps following the convention used in the harmonic map heat flow literature. Define the set of indices

$$\mathcal{J}_{\max} := \Big\{ j \in \{1, \ldots, M\} \mid C_j^{-1} \leq \frac{\lambda_{\max,n}}{\lambda(W_{j,n})} \leq C_j, \text{ for each } n \text{ for some } C_j > 1 \Big\}$$

and let K_0 be the smallest natural number such that

$$\sum_{j\in\mathcal{J}_{\max}} \bar{E}(\mathcal{W}_{j,0}) \in [K_0\bar{E}_*, (K_0 + 1)\bar{E}_*). \tag{3.10}$$

Then consider the following two cases.

Case 1: First, suppose that $K_0 = K$. Then $\mathcal{J}_{\max} = \mathcal{R} = \{1, \ldots, M\}$ and $M_j = 0$. The idea is that if one of the above conditions does not hold, then there exists a bubble which will cost at least $\bar{E}_*$ amount of energy. More concretely, using (3.8)

$$\begin{aligned} K\bar{E}_* \leq \sum_{j\in\mathcal{J}_{\max}} \bar{E}(\mathcal{W}_{j,0}) &\leq \sum_{j\in\mathcal{R}} \bar{E}(\mathcal{W}_{j,0}) \\ &\leq \sum_{j=1}^{M} \bar{E}(\mathcal{W}_{j,0}) + \sum_{j=1}^{M}\sum_{k=1}^{M_j} \bar{E}(\mathcal{W}_{j,k}) \\ &= \lim_{n\to\infty} \bar{E}(u(s_n); B(y_n, \rho_n)) < (K + 1)\bar{E}_*. \end{aligned}$$

From the above expression it is clear that if $j_0 \in \mathcal{R} \setminus \mathcal{J}_{\max}$ then $\bar{E}(\mathcal{W}_{j_0,0}) \geq \bar{E}_*$ which contradicts $\lim_{n\to\infty} \bar{E}(u(s_n); B(y_n, \rho_n)) < (K + 1)\bar{E}_*$. Therefore $\mathcal{J}_{\max} = \mathcal{R}$. By the same argument $\mathcal{R} = \{1, \ldots, M\}$ and $M_j = 0$ for each $j \in \{1, \ldots, M\}$. Therefore for each $j \in \{1, \ldots, M\}$

$$\lim_{n\to\infty} \bar{E}(W_{j,n} - \mathcal{W}_{j,0}\Big(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})}\Big); B(a(W_{j,n}),\ R_n\lambda(W_{j,n}))) = 0.$$

Fix a sequence $R_n \leq \breve{R}_n$ such that for each $j \in \{1, \ldots, M\}$ we have $4R_n\lambda_{\max,n} \leq \min_{j\in\{1,\ldots,M\}} \nu_{j,n}$. Then since $\lambda(W_{j,n}) \simeq \lambda_{\max,n}$ for each $j \in \{1, \ldots, M\}$ we can use (3.7) to get that

$$\lim_{n\to\infty} \bar{E}(u(s_n) - \mathcal{W}_{j,0}\Big(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})}\Big); B(a(W_{j,n}), 4R_n\lambda_{\max,n}))) = 0.$$

Now we can use Lemma 2.10 with $(t_n - s_n)^{1/2} \ll \lambda_{\max,n}$ to propagate these estimates to time t_n for each $j \in \{1, \ldots, M\}$ to get

$$\lim_{n\to\infty} \bar{E}(u(t_n) - \mathcal{W}_{j,0}\Big(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})}\Big); B(a(W_{j,n}), 4R_n\lambda_{\max,n}))) = 0. \tag{3.11}$$

The same reasoning applied to (3.8) yields

$$\lim_{n\to\infty} \bar{E}(u(t_n) - \mathcal{W}_{j,0}\Big(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})}\Big); B(y_n,\rho_n) \setminus \cup_{j=1}^{M} B(a(W_{j,n}); R_n\lambda_{\max,n}))) = 0. \quad (3.12)$$

Using (3.11), (3.12), pairwise disjointness of distinct balls $B(a(W_{j,n}), R_n\lambda(W_{j,n}))$, asymptotic orthogonality of the triples $(\mathcal{W}_{j,0}, a(W_{j,n}), \lambda(W_{j,n}))$, and Remark 3.2, we get that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(y_n,\rho_n)) = 0,$$

which contradicts the second equation in (3.6).

Case 2: Next, consider the case $K_0 < K$. We show that this case leads to a contradiction with the minimality of K. Again we will need $R_n \to \infty$ such that $4R_n\lambda_{\max,n} \le \min\{\nu_{j,n}\}_{j\in\mathcal{J}_{\max}}$ and $R_n \le \breve{R}_n$. We split the argument into several steps.

Step 1. We first show the existence of an integer $L \ge 1$, sequences $\{x_{\ell,n}\}_{\ell=1}^{L}$ with $x_{\ell,n} \in B(y_n,\xi_n)$ for each $n \in \mathbb{N}$ and each $\ell \in \{1,\dots,L\}$, and a sequence r_n satisfying the following properties

- $(t_n - s_n)^{1/2} \ll r_n \ll \lambda_{\max,n}$;
- the balls $B(x_{\ell,n}, r_n)$ are pairwise disjoint for $\ell \in \{1,\dots,L\}$ with

$$\lim_{n\to\infty} \frac{|x_{\ell,n} - x_{\ell',n}|}{r_n} = \infty \quad (3.13)$$

  for $\ell \ne \ell'$;
- on the union of all such balls, we capture the missing energy

$$\lim_{n\to\infty} \bar{E}(u(s_n); \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) \in [(K-K_0)\bar{E}_*, (K-K_0+1)\bar{E}_*), \quad (3.14)$$

  with vanishing energy in the neck region, i.e., there exist sequences $\alpha_n \to 0, \beta_n \to \infty$ such that

$$\lim_{n\to\infty} \sum_{\ell=1}^{L} \bar{E}(u(s_n); B(x_{\ell,n}, \beta_n r_n) \setminus B(x_{\ell,n}, \alpha_n r_n))) = 0; \quad (3.15)$$

- and a sequence $\breve{\xi}_n$ such that

$$\xi_n \ll \breve{\xi}_n \ll \rho_n, \quad B(x_{\ell,n}, \beta_n r_n) \subset B(y_n, \breve{\xi}_n). \quad (3.16)$$

Step 1.1. We first construct the sequence of points $\mathcal{P} := \{\{x_{\ell,n}\}_{\ell=1}^{L}\}$ for some integer $L \ge 1$. The idea will be to do this inductively. Define our initial set $\mathcal{P}_0$ to consist of all points such that

- $a(\mathcal{W}_{j,n})$ with $j \in \mathcal{R} \setminus \mathcal{J}_{\max}$,
- $a(\mathcal{W}_{j,n})$ with $\mathfrak{h}_j \in \mathcal{D}(j_0)$ for some $j_0 \in \mathcal{J}_{\max,n}$, where recall that $\mathcal{D}(j_0)$ is the collection of maximal elements in the pruned tree $\mathcal{T}(j_0) \setminus \mathfrak{h}_{j_0}$, and
- sequences $p_{j_0,k,n}$ associated to stationary solutions $\mathcal{W}_{j_0,k}\big(\frac{\cdot - p_{j_0,k,n}}{\Lambda_{j_0,k,n}}\big)$ for some $j_0 \in \mathcal{J}_{\max}$ that are
  - asymptotically orthogonal to every bubble in the collection $\mathfrak{h}_j \in \mathcal{D}(j_0)$,
  - and not children of any $\mathfrak{h}_j \in \mathcal{D}(j_0)$.

Enumerate the set of all such points, $\mathcal{P}_0 = \{\{y_{\ell,n}\}_{\ell=1}^{L'}\}$ for some integer $L' \ge 1$. Observe that after possibly passing to a subsequence we have

$$\lim_{n\to\infty} \frac{(t_n - s_n)^{1/2}}{\operatorname{dist}(y_{\ell,n}, y_{\ell',n})} \in [0,\infty]$$

for any $\ell \neq \ell' \in \{1, \ldots, L'\}$. We add $y_{\ell_0,n}$ to our final collection $\mathcal{P}$ if

$$\lim_{n\to\infty} \frac{(t_n - s_n)^{1/2}}{\operatorname{dist}(y_{\ell_0,n}, y_{\ell,n})} = 0, \quad \forall \ell \in \{1, \ldots, L')\} \setminus \ell_0.$$

Otherwise, denote

$$\mathcal{D}(\ell_0) := \{\ell_0\} \cup \Big\{\ell : \lim_{n\to\infty} \frac{(t_n - s_n)^{1/2}}{\operatorname{dist}(y_{\ell_0,n}, y_{\ell,n})} > 0\Big\}.$$

Note that $\mathcal{D}(\ell_1) = \mathcal{D}(\ell_2)$ iff $\ell_2 \in \mathcal{D}(\ell_1)$. Define the barycenter

$$x_{\ell_0,n} := \sum_{\ell \in \mathcal{B}(\ell_0)} \frac{y_{\ell,n}}{|\mathcal{B}(\ell_0)|}.$$

We will include $x_{\ell_0,n} \in \mathcal{P}$. This finishes the construction of the set $\mathcal{P} = \{\{x_{\ell,n}\}_{\ell=1}^{L}\}$ for some integer $L \geq 1$ such that $\{x_{\ell,n}\} \subset B(y_n, \xi_n)$ for any $1 \leq \ell \leq L$.
Step 1.2. We choose the scale r_n such that

$$\begin{aligned} (t_n - s_n)^{1/2} \ll r_n \ll \lambda_{\max,n}, \quad \max\{R_n\lambda(W_{j,n}), \nu_{j,n}\} \ll r_n, \quad \forall j \notin \mathcal{J}_{\max} \\ \max(\Lambda_{j,k,n}, \xi_{j,n}) \ll r_n, \quad \forall\, (j,k) \in \mathcal{J}_{\max} \times \{1, \ldots, M_j\} \end{aligned}$$

and such that the balls $B(x_{\ell,n}, r_n)$ satisfy (3.13). Note that $B(x_{\ell,n}, r_n)$ is asymptotically disjoint from $B(a(W_{j_0,n}), R_n\lambda_{\max,n})$ for any $j_0 \in \mathcal{J}_{\max}$ since $\lambda_{\max,n}^{-1}|a(W_{j_0,n}) - a(W_{j,n})| \to \infty$ for all $j \in \mathcal{R} \setminus \mathcal{J}_{\max}$ and $j_0 \in \mathcal{J}_{\max}$ and we choose the points $x_{\ell,n}$ to be coming from the centers $a(W_{j,n})$ for $j \in \mathcal{R} \setminus \mathcal{J}_{\max}$. This concludes the construction of the centers $\{x_{\ell,n}\}$ and scales r_n.
Step 1.3. It remains to verify (3.14), (3.15), and (3.16). The construction of the sequence $\breve{\xi}_n$ such that (3.16) holds follows from the construction of the scales r_n. For the other two estimates, observe that for any $j_0 \in \mathcal{J}_{\max}$, by definition $\{x_{\ell,n}\}_{\ell=1}^{L}$, the limit in (3.9), and the choice of r_n we have

$$\lim_{n\to\infty} \bar{E}(u(s_n) - \tilde{\mathcal{W}}_{j_0,0}; B(a(W_{j_0,n}), 4R_n\lambda_{\max,n}) \setminus \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n))) = 0, \tag{3.17}$$

where we define $\tilde{\mathcal{W}}_{j_0,0} := \mathcal{W}_{j_0,0}\big(\frac{\cdot - a(W_{j_0,n})}{\lambda(W_{j_0,n})}\big)$. Since $r_n \ll \lambda_{\max,n}$, the stationary solution

$$\lim_{n\to\infty} \bar{E}(\tilde{\mathcal{W}}_{j_0,0}; \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) = 0. \tag{3.18}$$

Equations (3.18), (3.17), (3.10), and (3.8) imply that

$$\lim_{n\to\infty} \bar{E}(u(s_n); B(y_n, \rho_n) \setminus \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) = \sum_{j \in \mathcal{J}_{\max}} \bar{E}(\mathcal{W}_{j,0}) \in [K_0\bar{E}_*, (K_0+1)\bar{E}_*). \tag{3.19}$$

Then,

$$\begin{aligned} \lim_{n\to\infty} \bar{E}(u(s_n); \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) &= \lim_{n\to\infty} \bar{E}(u(s_n); B(y_n, \rho_n)) - \sum_{j \in \mathcal{J}_{\max}} \bar{E}(\mathcal{W}_{j,0}) \\ &\in ((K - K_0 - 1)\bar{E}_*, (K - K_0 + 1)\bar{E}_*). \end{aligned}$$

Since each bubble contributes atleast $\bar{E}_*$ amount of energy $\lim_{n\to\infty} \bar{E}(u(s_n); \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n))) \notin ((K - K_0 - 1)\bar{E}_*, (K - K_0)\bar{E}_*)$. Thus we must have that $\lim_{n\to\infty} \bar{E}(u(s_n); \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n))) \in [(K-K_0)\bar{E}_*, (K-K_0+1)\bar{E}_*)$ verifying (3.14). The condition (3.15) follows from the construction of the set $\mathcal{P}$ and the choice of r_n.
Step 2. The key point of constructing the collection of balls, $B(x_{\ell,n}, r_n)$ for $1 \leq \ell \leq L$, is that for large enough n, the function $u(t_n)$ deviates from a multi-bubble configuration on at least one

of these balls. In other words, we will now show that there exists $1 \le \ell_1 \le L$ and $\eta_1 > 0$ such that (after possibly passing to a subsequence)

$$\boldsymbol{\delta}(u(t_n); B(x_{\ell_1,n}, r_n)) \ge \eta_1. \tag{3.20}$$

If not then for all $\ell \in \{1, \dots, L\}$ we have

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(x_{\ell,n}, r_n)) = 0. \tag{3.21}$$

We will argue that this implies that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(y_n, \rho_n)) = 0,$$

which contradicts (3.6). First, since $(t_n - s_n)^{1/2} \ll r_n$ we can use Lemma 2.10 to propagate (3.15), (3.14), (3.19), and (3.17) up to time t_n to get

$$\begin{aligned}
&\lim_{n\to\infty} \bar{E}(u(t_n); \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) \in [K_1\bar{E}_*, (K_1+1)\bar{E}_*),\\
&\lim_{n\to\infty} \bar{E}(u(t_n); B(y_n, \rho_n) \setminus \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) \in [K_0\bar{E}_*, (K_0+1)\bar{E}_*), \text{ and}\\
&\lim_{n\to\infty} \bar{E}(u(t_n) - \tilde{W}_{j_0,0}; B(a(W_{j_0,n}), R_n\lambda_{\max,n}) \setminus \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n)) = 0,
\end{aligned} \tag{3.22}$$

where $K_1 = K - K_0$, $j_0 \in \mathcal{J}_{\max}$. Using again Lemma 2.10, (3.8), the construction of the sequences $\{x_{\ell,n}\}$ and r_n we have

$$\lim_{n\to\infty} \bar{E}(u(t_n); B(y_n, \rho_n) \setminus (\cup_{j\in\mathcal{J}_{\max}} B(a(W_{j,n}), R_n\lambda_{\max,n}) \cup \cup_{\ell=1}^{L} B(x_{\ell,n}, r_n))) = 0. \tag{3.23}$$

From (3.21), after passing to a joint subsequence in n, for each $\ell \in \{1, \dots, L\}$ we can find an integer $\widetilde{M}_\ell \ge 0$, a sequence of $\widetilde{M}_\ell$-bubble configurations $\mathbf{W}(\vec{W}_{\ell,n})$, and sequences of vectors $\vec{\nu}_{\ell,n} = (\nu_{\ell,n}, \nu_{\ell,1,n}, \dots, \nu_{\ell,\widetilde{M}_\ell,n})$ and $\vec{\xi}_{\ell,n} = (\xi_{\ell,n}, \xi_{\ell,1,n}, \dots, \xi_{\ell,\widetilde{M}_\ell,n})$, so that

$$\lim_{n\to\infty} \mathbf{d}(u(t_n), \mathbf{W}(\vec{W}_{\ell,n}); B(x_{\ell,n}, r_n); \vec{\nu}_{\ell,n}, \vec{\xi}_{\ell,n}) = 0. \tag{3.24}$$

Here note that $\mathbf{W}(\vec{W}_{\ell,n}) = \sum_{j=1}^{\widetilde{M}_\ell} W_{\ell,j,n}$ for some collection of stationary solutions $W_{\ell,j,n}$. Consider collection of maps

$$\widetilde{W}_n = ((W_{\ell,j,n})_{\ell=1,j=1}^{L,\widetilde{M}_\ell}, (W_{j,0,n})_{j\in\mathcal{J}_{\max}})$$

where $W_{j,0,n} := \mathcal{W}_{j,0}(\frac{\cdot - a(W_{j,n})}{\lambda(W_{j,n})})$ are the weak limits obtained by applying the compactness Theorem 2.15 for each $j \in \mathcal{J}_{\max}$ and let $\mathbf{W}(\vec{\widetilde{W}}_n)$ denote the sum of all the maps in the above collection. For each $j \in \mathcal{J}_{\max}$ set $\nu_{j,n} := R_n$, $\xi_{j,n} = r_n$ and

$$\vec{\tilde{\nu}}_n := (\nu_n, (\nu_{\ell,n})_{\ell=1}^{L}, (\nu_{j,n})_{j\in\mathcal{J}_{\max}}), \quad \vec{\tilde{\xi}}_n := (\breve{\xi}_n, (\xi_{\ell,n})_{\ell=1}^{L}, (\xi_{j,n})_{j\in\mathcal{J}_{\max}}).$$

Then we claim that

$$\lim_{n\to\infty} \mathbf{d}\big(u(t_n), \mathbf{W}(\vec{\widetilde{W}}_n); B(y_n, \rho_n); \vec{\tilde{\nu}}_n, \vec{\tilde{\xi}}_n\big) = 0. \tag{3.25}$$

This follows from (3.24), asymptotic orthogonality of distinct triples $(W_{\ell,k,n}, a(W_{\ell,k,n}), \lambda(W_{\ell,k,n}))$ and $(W_{\ell',k',n}, a(W_{\ell',k',n}), \lambda(W_{\ell',k',n}))$ for $(\ell,k) \ne (\ell',k')$ since $B(x_{\ell,n}, r_n)$ are mutually disjoint, asymptotic orthogonality of triples

$$(W_{\ell,k,n}, a(W_{\ell,k,n}), \lambda(W_{\ell,k,n})) \quad \text{and} \quad \Big(W_{j_0,0}(\frac{\cdot - a(W_{j_0,n})}{\lambda(W_{j_0,n})}), a(W_{j_0,n}), \lambda(W_{j_0,n})\Big)$$

for any $1 \le \ell \le L$ and $j_0 \in \mathcal{J}_{\max}$ since $r_n \ll \lambda_{\max,n}$ and

$$\lim_{n\to\infty} \bar{E}(\tilde{W}_{j_0,0}; B(x_{\ell,n}, r_n))) = 0, \quad \forall j_0 \in \mathcal{J}_{\max}, \quad \forall \ell \in \{1, \dots, L\},$$

$$\lim_{n\to\infty} \bar{E}(W_{\ell,k,n}; B(y_n, \rho_n) \setminus B(x_{\ell,n}, r_n))) = 0, \quad \forall\, \ell \in \{1, \dots, L\},\, k \in \{1, \dots, \widetilde{M}_\ell\}.$$

These observations, together with (3.22), (3.23), and Remark 3.2 applied with scale $\check{\xi}_n$, yields (3.25). This establishes (3.20).

Step 3. As a consequence of (3.20) we will show that there exists $\widetilde{\sigma}_n < t_n$ such that

$$t_n - \widetilde{\sigma}_n \ll r_n^2 \ \text{ and } \ \lim_{n\to\infty} r_n \|\mathcal{T}(u(\widetilde{\sigma}_n))\|_{L^2} = 0. \tag{3.26}$$

This follows from the same contradiction argument as in the Proof of Lemma 3.3. Applying Theorem 2.15 and possibly passing to a subsequence, we have a bubble tree decomposition as in (2.24) for some sequence $\hat{R}_n \to \infty$. The estimate (3.15) can be propagated to time $\widetilde{\sigma}_n$ using (2.7) and the argument in Lemma 2.10 to get

$$\lim_{n\to\infty} \bar{E}\big(u(\widetilde{\sigma}_n); B(x_{\ell,n}, \beta_n r_n/2) \setminus B(x_{\ell,n}, 2\alpha_n r_n)\big) = 0. \tag{3.27}$$

Therefore, all the stationary solutions at scale r_n in Theorem 2.15 vanish, which implies that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(\widetilde{\sigma}_n); B(x_{\ell_1,n}, r_n)) = 0. \tag{3.28}$$

By (2.27) we can find an integer $K' \ge 0$ so that,

$$\bar{E}(u(\widetilde{\sigma}_n); B(x_{\ell_1,n}, r_n))) \in [K'\bar{E}_*, (K'+1)\bar{E}_*) \ \text{ as } \ n \to \infty. \tag{3.29}$$

The estimate (3.20) implies that $K' \ge 1$ since $t_n - \widetilde{\sigma}_n \ll r_n^2$.

Step 4. We will now show that $K' < K$ and that $[\widetilde{\sigma}_n, t_n] \in \mathcal{C}_{K'}(x_{\ell_1,n}, r_n, \varepsilon_{1,n}, \eta_1)$ for some sequence $\varepsilon_{1,n} \to 0$ which will contradict the minimality of K.

Step 4.1. We first show that $K' < K$. When $K_0 > 0$, $K' < K$ since some energy lives on the scale comparable to the maximum scale $\lambda_{\max,n}$ which is asymptotically larger than r_n. On the other hand, suppose $K_0 = 0$. If $K' = K$ then this implies that the energy in $B(y_n, \rho_n)$ is successfully captured by the balls $B(x_{\ell_1,n}, r_n)$. However, since there is at least one index j_0 attaining the maximum scale, i.e., $\lambda(W_{j_0,n}) = \lambda_{\max,n}$, and $r_n \ll \lambda_{\max,n} = \lambda(W_{j_0,n})$, by the Definition of the scale 1.1, we see that at least $\bar{E}_*/2$ energy must live outside the scale $B(x_{\ell_1,n}, r_n)$ which is a contradiction to (3.27). Thus $K' < K$.

Step 4.2. Next, we check the properties of the Definition 3.1. Item (1) follows from (3.28), item (2) follows from (3.20), item (3) follows from (3.26) and item (4) follows from (3.29). Thus,

$$[\widetilde{\sigma}_n, t_n] \in \mathcal{C}_{K'}(x_{\ell_1,n}, r_n, \varepsilon_{1,n}, \eta_1)$$

which is a contradiction to the minimality of K, and therefore the proof is complete. □

By a standard continuity argument, we get the following Corollary of the above Lemma.

Corollary 3.8. *Let $\eta_0 > 0$ be as in Lemma 3.4, $\eta \in (0, \eta_0]$, and $[\sigma_n, \tau_n] \in \mathcal{C}_K(y_n, \rho_n, \varepsilon_n, \eta)$. Then, there exist $\varepsilon \in (0, \eta)$, $c_0 > 0$ $n_0 \in \mathbb{N}$, and $s_n \in (\sigma_n, \tau_n)$ such that for all $n \ge n_0$, the following conclusions hold. First,*

$$\boldsymbol{\delta}(u(s_n); B(y_n, \rho_n)) = \varepsilon.$$

Moreover, for each $n \ge n_0$ let $M_n \in \mathbb{N}$, and $\mathbf{W}(\vec{W}_n)$, where $\vec{W}_n = (W_1, \dots W_{M_n})$ be any sequence of M_n-bubble configurations, and let $\vec{\nu}_n = (\nu_n, \nu_{1,n}, \dots, \nu_{M,n}), \vec{\xi}_n = (\xi_n, \xi_{1,n}, \dots, \xi_{M,n}) \in (0,\infty)^{M+1}$ be any admissible sequences in the sense of Definition 1.4 such that

$$\varepsilon \le \mathbf{d}(u(s_n), \mathbf{W}(\vec{W}_n); B(y_n, \rho_n), \vec{\nu}_n, \vec{\xi}_n) \le 2\varepsilon$$

for each n. Define

$$\lambda_{\max,n} = \lambda_{\max}(s_n) := \max_{j=1,\dots,M_n} \lambda(W_{j,n}).$$

Then, $s_n + c_0\lambda_{\max}(s_n)^2 \le \tau_n$ and,

$$\boldsymbol{\delta}(u(t); B(y_n, \rho_n)) \ge \varepsilon, \quad \forall\, t \in [s_n, s_n + c_0\lambda_{\max}(s_n)^2].$$

Proof. From Lemma 3.4, fix $\varepsilon, \eta_0 > 0$. Then we can define s_n by the first exit time

$$s_n := \inf\{t \in [\sigma_n, \tau_n] \mid \boldsymbol{\delta}(u(\tau); B(y_n, \rho_n)) \ge \varepsilon, \text{ for all } \tau \in [t, \tau_n]\}.$$

This is well-defined for all sufficiently large n. Then by continuity, $\boldsymbol{\delta}(u(s_n); B(y_n, \rho_n)) = \varepsilon$. Setting $\lambda_{\max}(s_n)$ and using Lemma 3.4 we see that for n large enough we have

$$s_n + c_0\lambda_{\max}(s_n)^2 \le \tau_n,$$

which completes the proof. □

4. Conclusion

In this section, we will prove Theorem 1.8 and use it to establish Theorem 1.6 and Corollary 1.7.

Proof of Theorem 1.8. The proof proceeds by a contradiction argument that we break into several steps.
Step 1. Setting up the contradiction hypothesis. If Theorem 1.8 fails then there exists a non-negative integer $K \ge 1$, and parameters

$$y_n \in \mathbb{R}^d,\ \rho_n > 0,\ 0 < \sigma_n < \tau_n < T_+, \quad [\,\sigma_n, \tau_n\,] \in \mathcal{C}_K(y_n, \rho_n, \varepsilon_n, \eta),$$

with $\varepsilon_n \to 0$, $\sigma_n, \tau_n \to T_+$ such that

$$|\,\tau_n - \sigma_n\,| \le \varepsilon_n \rho_n^2, \quad \boldsymbol{\delta}\big(u(\sigma_n); B(y_n, \rho_n)\big) \le \varepsilon_n, \quad \boldsymbol{\delta}\big(u(\tau_n); B(y_n, \rho_n)\big) \ge \eta,$$

and $\bar{E}\big(u(\sigma_n); B(y_n, \rho_n)\big) \in [K\bar{E}_*, (K+1)\bar{E}_*)$.
Step 2. Picking the first exit time inside each collision interval. By Corollary 3.8 there exist $\varepsilon \in (0, \eta)$, $c_0 > 0$, and times

$$s_n \in (\sigma_n, \tau_n), \quad \boldsymbol{\delta}\big(u(s_n); B(y_n, \rho_n)\big) = \varepsilon,$$

such that for $s_n + c_0\lambda_{\max,n}^2 \le \tau_n$ and for all $t \in [s_n, s_n + c_0\lambda_{\max,n}^2]$ we have

$$\boldsymbol{\delta}(u(t); B(y_n, \rho_n)) \ge \varepsilon \tag{4.1}$$

where $\lambda_{\max,n} := \lambda_{\max}(s_n)$.
Step 3. A quantitative lower bound on the $\|\partial_t u(t)\|_{L^2}^2$. We claim that there exists a constant $c_1 > 0$ such that for n large enough we have,

$$\lambda_{\max,n}^2 \,\|\partial_t u(t)\|_{L^2}^2 \ \ge\ c_1, \quad \forall t \in [s_n, s_n + c_0\lambda_{\max,n}^2]. \tag{4.2}$$

We will prove this by contradiction.
Step 3.1. Setting up the contradiction hypothesis. If (4.2) does not hold then there exists a sequence of times $t_n \in [s_n, s_n + c_0\lambda_{\max,n}^2]$ such that

$$\lambda_{\max,n} \,\|\partial_t u(t_n)\|_{L^2} \to 0$$

as $n \to \infty$. Using Theorem 2.15, we deduce that (up to a subsequence) there exists $R_n(x_n) \to \infty$ such that for any sequence $1 \ll \check{R}_n \ll R_n(x_n)$ we have

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(x_n, \check{R}_n\lambda_{\max,n})) = 0. \tag{4.3}$$

We will construct a set of points $x_{\ell,n}$ for $1 \leq \ell \leq L$ for some integer $L \geq 1$ and use (4.3) to conclude that

$$\lim_{n\to\infty} \boldsymbol{\delta}\big(u(t_n); B(y_n,\rho_n)\big) = 0. \tag{4.4}$$

which will contradict the lower bound $\boldsymbol{\delta}(u(t_n); B(y_n,\rho_n)) \geq \varepsilon$.

Step 3.2. Construction of the sequence $\{x_{\ell,n}\}_{\ell=1}^{L}$. We claim that there exist an integer $L \geq 1$, points $\{x_{\ell,n}\}$ for $1 \leq \ell \leq L$, $R \geq 2$ and a sequence $1 \ll \widetilde{R}_n \ll \lambda_{\max,n}^{-1}\xi_n$ such that

$$\bar{E}(u(s_n); B(y_n,\rho_n) \setminus \cup_{\ell=1}^{L} B(x_{\ell,n}, R\lambda_{\max,n})) \leq \frac{\bar{E}_*}{4}, \tag{4.5}$$

$$B(x_{\ell,n}, \widetilde{R}_n\lambda_{\max,n}) \cap B(x_{\ell',n}, \widetilde{R}_n\lambda_{\max,n}) = \emptyset, \forall \ell \neq \ell' \in \{1,\ldots,L\}, \tag{4.6}$$

where K is defined in Step 1, ξ_n comes from the multi-bubble configuration obtained at $t = s_n$, i.e., we consider multi-bubble configurations $\mathbf{W}(\vec{W}_n) = \sum_{j=1}^{M} W_{j,n}$ comprising of some fixed M number of bubbles, after possibly passing to a subsequence because our solution has finite energy with parameters $\vec{\nu}_n$ and $\vec{\xi}_n$ such that

$$\varepsilon \leq \mathbf{d}(u(s_n), \mathbf{W}(\vec{W}_n); B(y_n,\rho_n); \vec{\nu}_n, \vec{\xi}_n) \leq 2\varepsilon. \tag{4.7}$$

We define ξ_n, ν_n as the first components of the vectors $\vec{\xi}_n, \vec{\nu}_n$ respectively. Arguing as in Remark 3.2, we deduce that (3.2) and (3.3) hold. We will construct the sequence $\{x_{\ell,n}\}_{\ell=1}^{L}$ for some integer $L \in \mathbb{N}$ as follows. First up, to a subsequence we have that

$$L_{jk} := \lim_{n\to\infty} \frac{|a(W_{j,n}) - a(W_{k,n})|}{\lambda_{\max,n}} \in [0,\infty], \quad \forall j \neq k \in \{1,\ldots,M\}.$$

Given an index $j \in \{1,\ldots,M\}$ we collect all other indices for which L_{jk} is finite, i.e.,

$$\mathcal{L}(j) := \{j\} \cup \Big\{k \in \{1,\ldots,M\} : L_{jk} < \infty\Big\}.$$

Observe that for $j \neq k$, either $\mathcal{L}(j) = \mathcal{L}(k)$ or $\mathcal{L}(j) \cap \mathcal{L}(k) = \emptyset$. Define the barycenter

$$x_{\mathcal{L}(j),n} := \sum_{i\in\mathcal{L}(j)} \frac{a(W_{i,n})}{|\mathcal{L}(j)|}.$$

Then our desired sequence of points $\{x_{\ell,n}\}_{\ell=1}^{L}$ is simply a collection of points $\{x_{\mathcal{L}(j),n}\}$ for each distinct index set $\mathcal{L}(j)$ with $L \leq M$.

Step 3.3 Verification of (4.5) and (4.6). Using Lemma 2.2, (4.7), and the definitions of $\mathbf{d}$ and $\lambda_{\max,n}$ there exists $R_1 \gg 1$ such that for $n \gg 1$ we have

$$E\Big(u(s_n); B(y_n,\rho_n) \setminus \bigcup_{j=1}^{M} B(a(\omega_{j,n}), R_1\lambda_{\max,n})\Big) \leq \frac{\bar{E}_*}{4},$$

where K is defined in Step 1. Then, the definition of the points $x_{\ell,n}$ yields a sequence $1 \ll \widetilde{R}_n \ll \lambda_{\max,n}^{-1}\xi_n$ such that (4.6) holds.

Step 4. Vanishing of the distance in (4.4). Using the collection $\{x_{\ell,n}\}_{\ell=1}^{L}$ as the centers in (4.3) consider sequences $R_{\ell,n}$ such that for any $\breve{R}_n \leq R_{\ell,n}$ we have

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(x_{\ell,n}, \breve{R}_n\lambda_{\max,n})) = 0, \quad \ell \in \{1,\ldots,L\}. \tag{4.8}$$

This in particular implies that there exists an integer $K_\ell \geq 0$ such that

$$\lim_{n\to\infty} \bar{E}(u(t_n); B(x_{\ell,n}, R_n\lambda_{\max,n})) \in [K_\ell\bar{E}_*, (K_\ell+1)\bar{E}_*)$$

for each $\ell \in \{1, \dots, L\}$. Now consider $\widetilde{\xi}_n$ such that $\xi_n \ll \widetilde{\xi}_n \ll \rho_n$. Then, for each $\ell \in \{1, \dots, L\}$ we have

$$\lim_{n\to\infty} \frac{\lambda_{\max,n}}{\operatorname{dist}(x_{\ell,n}, \partial B(y_n, \widetilde{\xi}_n))} = 0, \tag{4.9}$$

as $x_{\ell,n} \in B(y_n, \xi_n)$ and $\lambda_{\max,n} \ll \xi_n$. Therefore, there exists a sequence $R_n \le \min\{\widetilde{R}_n, \{R_{\ell,n}\}_{\ell=1}^L\}$ such that $B(x_{\ell,n}, R_n\lambda_{\max,n}) \subset B(y_n, \widetilde{\xi}_n)$ for each $\ell \in \{1, \dots, L\}$. Thus

$$\bar{E}(u(s_n); B(y_n, \rho_n) \setminus \cup_{\ell=1}^L B(x_{\ell,n}, R_n\lambda_{\max,n}/2)) \le \frac{\bar{E}_*}{4}.$$

Propagating the above estimate using Lemma 2.10 we get

$$\lim_{n\to\infty} \bar{E}(u(t_n); B(y_n, \rho_n) \setminus \cup_{\ell=1}^L B(x_{\ell,n}, R_n\lambda_{\max,n})) \le \frac{\bar{E}_*}{2}. \tag{4.10}$$

Using (4.3) for points in $\Omega_{n,L} := B(y_n, \rho_n) \setminus \cup_{\ell=1}^L B(x_{\ell,n}, R_n\lambda_{\max,n})$ we deduce that $u(t_n)$ cannot be close to a single bubble due to (4.10) and therefore

$$\lim_{n\to\infty} \bar{E}(u(t_n); \Omega_{n,L}) = 0.$$

We also know that (4.8) and the definition of the sequence R_n implies

$$\lim_{n\to\infty} \sum_{\ell=1}^{L} \boldsymbol{\delta}(u(t_n); B(x_{\ell,n}, R_n\lambda_{\max,n})) = 0. \tag{4.11}$$

Moreover, the balls $B(x_{\ell,n}, R_n\lambda_{\max,n})$ are disjoint by (4.6) and the choice of $R_n \le \widetilde{R}_n$. Combining (4.11), (4.9), the disjointness of the balls $B(x_{\ell,n}, R_n\lambda_{\max,n})$, (4.10), and Remark 3.2, we conclude that

$$\lim_{n\to\infty} \boldsymbol{\delta}(u(t_n); B(y_n, \rho_n)) = 0,$$

which contradicts (4.1). Thus (4.2) holds.
Step 5. Conclusion. By (4.2) we have

$$\begin{aligned}\int_0^{T_+} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t &\ge \sum_n \int_{s_n}^{s_n + c_0\lambda_{\max}(s_n)^2} \|\partial_t u(t)\|_{L^2}^2 \, \mathrm{d}t \\ &\ge c_1 \sum_n \int_{s_n}^{s_n + c_0\lambda_{\max}(s_n)^2} \lambda_{\max}(s_n)^{-2} \, \mathrm{d}t \ge c_0 c_1 \sum_n 1 = \infty,\end{aligned}$$

which contradicts (1.3). Thus, we have proved Theorem 1.6. □

Proof of Theorem 1.6. We treat the finite–time blow–up case $T_+ < \infty$; the global case is analogous. Throughout we write $\rho(t) := \sqrt{T_+ - t}$.
Step 1. Reduction to small balls near the bubbling points. Theorem 2.8 furnishes the existence of the set $\{x_1, \dots, x_L\} \subset \mathbb{R}^d$ and a weak limit $u_* \in \dot{H}^1$. Choosing $0 < \rho_0 \ll 1$ so that the balls $B(x_\ell, 2\rho_0)$ are disjoint, Theorem 2.8 implies

$$\lim_{t\to T_+} \bar{E}(u(t) - u_*;\, \mathbb{R}^d \setminus \cup_{\ell=1}^L B(x_\ell, \rho_0)) = 0 \quad \text{and} \quad \lim_{t\to T_+} \bar{E}(u(t) - u_*;\, B(x_\ell, \rho_0) \setminus B(x_\ell, \rho(t))\Big) = 0$$

for $1 \le \ell \le L$. Since $u_* \in \dot{H}^1$, we have $\bar{E}\big(u_*; B(x_\ell, \rho(t))\big) \to 0$ as $t \to T_+$. Hence it suffices to study $u(t)$ inside the shrinking balls $B(x_\ell, \rho(t))$.

Step 2. Bubbling at one blowup point. Fix one bubbling point and denote it by $y := x_\ell$. Theorem 1.8 gives

$$\lim_{t\to T_+} \boldsymbol{\delta}(u(t); B(y,\rho(t))) = 0.$$

Let $t_n \to T_+$ be an arbitrary sequence of times. Then there exist

- an integer $M_n \geq 1$, which is also finite since $u(t)$ is a finite energy solution;
- $\mathbf{W}(\vec{W}_n) = \sum_{j=1}^{M_n} W_{j,n}$, where $W_{j,n}$ are stationary solutions;
- and scales $\vec{\nu}_n = (\nu_{0,n}, \nu_{1,n}, \ldots, \nu_{M_n,n})$ and $\vec{\xi}_n = (\xi_{0,n}, \xi_{1,n}, \ldots, \xi_{M_n,n})$,

such that

$$\mathbf{d}(u(t_n), \mathbf{W}(\vec{W}_n), ; B(y, \rho(t_n)), \vec{\nu}_n, \vec{\xi}_n) \to 0 \quad \text{as } n \to \infty.$$

Upon passing to a subsequence, we may assume that $M_n = M$ for all n.

Step 2.1 Initial bubble tree construction. For every $j \in \{1, \ldots, M\}$ the map $W_{j,n}$ is a stationary solution and therefore applying Theorem 2.15 we get for each fixed j

- an integer $M_j \geq 0$;
- a weak limit $\vartheta_{j,0}$;
- non-zero stationary solutions $\vartheta_{j,1}, \ldots, \vartheta_{j,M_j}$ with center $p_{j,k,n} \in B(a(W_{j,n}), \lambda(W_{j,n}))$ and scales $\Lambda_{j,k,n} \ll \lambda(W_{j,n})$,

such that

$$\lim_{n\to\infty} \bar{E}\Big(W_{j,n} - \vartheta_{j,0}\big(\frac{\cdot - a(W_{j,})}{\lambda(W_{j,n})}\big) - \sum_{k=1}^{M_j} \vartheta_{j,k}\big(\frac{\cdot - p_{j,k,n}}{\Lambda_{j,k,n}}\big); B_{j,n}\Big) = 0,$$

where $B_{j,n} := B(a(W_{j,n}),\, R_n\lambda(W_{j,n}))$ for some $R_n \to \infty$, where the scales and centers satisfy

$$\lim_{n\to\infty} \sum_{k\neq k'} \Big(\frac{\Lambda_{j,k,n}}{\Lambda_{j,k',n}} + \frac{\Lambda_{j,k',n}}{\Lambda_{j,k,n}} + \frac{|p_{j,k,n} - p_{j,k',n}|^2}{\Lambda_{j,k,n}\Lambda_{j,k',n}}\Big)^{-1} = 0.$$

For convenience denote $\Lambda_{j,0,n} := \lambda(W_{j,n}), p_{j,0,n} := a(W_{j,n})$ so that in every bubble family indexed by (j,k) the index $k = 0$ corresponds to the original scale $\lambda(W_{j,n})$ and centre $a(W_{j,n})$.

Step 3. Refined bubble tree construction. By the construction in the previous step, we have found a family

$$\{(\vartheta_{j,k}, p_{j,k,n}, \Lambda_{j,k,n})\}_{j=1,k=0}^{j=M,k=M_j}$$

which looks promising, but unfortunately, might not be asymptotically orthogonal. However, we can follow the same argument as in the proof of Theorem 1 in [JLS25] to construct an asymptotic orthogonal family $(W_j, a_{j,n}, \lambda_{j,n})$. The idea is to analyze the bubble tree as in the proof of Lemma 3.4. Denote $\mathcal{R}$ to be set of root indices obtained after partially ordering the tree $\mathfrak{h}_j = \{W_{j,n}\}_{n=1}^{\infty}$ and for each $\mathfrak{h}_{j_0} \in \mathcal{R}$ consider the bubble tree $\mathcal{T}(j_0) := \{\mathfrak{h}_j \preceq \mathfrak{h}_{j_0}\}$. For some large constant $C' > 0$, $B(a(W_{j,n}), \lambda(W_{j,n})) \subset B(a(W_{j_0,n}), C'\lambda(W_{j_0,n}))$ and therefore the domain $B(a(W_{j_0,n}), C'\lambda(W_{j_0,n}))$ contains all the stationary solutions

$$\bigcup_{\mathfrak{h}_j \in \mathcal{T}(j_0)} \{(\vartheta_{j,k}, p_{j,k,n}, \Lambda_{j,k,n})\}_{k=0}^{M_j}.$$

We will refine this collection to obtain an asymptotic orthogonal family. To this end, define

$$\mathcal{K}(j,k) := \{(j,k)\} \cup \big\{(j',k') : (W_{j',k'}, p_{j',k',n}, \Lambda_{j',k',n}) \perp (W_{j,k}, p_{j,k,n}, \Lambda_{j,k,n})\big\}.$$

For each reference index $j_0 \in \mathcal{R}$ we examine every cluster $\mathcal{K}(j,k)$ attached to the preliminary list of triples $\big(\vartheta_{j,k}, p_{j,k,n}, \Lambda_{j,k,n}\big)$.

- *Case 1*: $\big|\mathcal{K}(j,k)\big| = 1$: we keep the lone triple $\big(\vartheta_{j,k}, p_{j,k,n}, \Lambda_{j,k,n}\big)$.

- *Case 2:* $\left|\mathcal{K}(j,k)\right| > 1$: discard *all* triples with first index in $\mathcal{K}(j,k)$ and or replace them by a single triple $\left(\Theta_{j,k}, p_{j,k,n}, \Lambda_{j,k,n}\right)$, where $\Theta_{j,k}$ is a stationary solution. The construction of this new bubble $\Theta_{j,k,n}$ uses Theorem 1.6 and Theorem 2.15, and therefore the argument from the proof of Theorem 1 in [JLS25] carries over to this setting as well.

Repeating this procedure for every root index $j_0 \in \mathcal{R}$ leaves a final family of triples that are pairwise asymptotically orthogonal and fulfill the conclusions of Theorem 1.6. □

Proof of Corollary 1.7. Since $u_0 \geq 0$, then by the maximum principle $u(t) \geq 0$ for all $t \in [0, T_+)$. By [CGS89] all the positive bubbles are classified and are up to scaling and translation of the form

$$W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{(d-2)}{2}}.$$

Modifying the definition of the localized distance 1.4 by only considering positive bubbles, one can repeat the argument in Section 3, proof of Theorem 1.8 and Theorem 1.6 to deduce the (1.6) and (1.8) with the solitons being independent of the sequence of times. The key point in these lemmas is the application of the Elliptic Compactness Theorem 2.15, which will produce positive bubbles given a positive sequence of finite energy functions. Furthermore, since the energy of positive bubbles is quantized, in item (4) of Definition 1.4 it suffices to simply redefine K as follows

$$\lim_{n\to\infty} \bar{E}(u(\sigma_n); B(y_n, \rho_n)) = K\bar{E}_*$$

for some integer $K \geq 0$. As a result, these modifications allow us to prove the Soliton Resolution Conjecture for the energy-critical nonlinear heat flow with non-negative initial data. □

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